On the Centre of Strong Graded Monads

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Abstract. We introduce the notion of "centre" for pomonoid-graded strong monads which generalizes some previous work that describes the centre of (not graded) strong monads. We show that, whenever the centre exists, this determines a pomonoid-graded commutative submonad of the original one. We also discuss how this relates to duoidally-graded strong monads.

Keywords: Category Theory · Graded Monad · Centre.

1 Introduction and Related Work

The notions of *centre/centrality* and similarly *commutant/centraliser* can be formulated for many different kinds of algebraic structures, e.g. monoids, groups, semirings. It also makes sense in certain kinds of categorical settings. For instance, premonoidal categories admit a centre [9], which is essential for the development of the theory. Another example is given by enriched algebraic theories and associated monads which were shown to admit centralisers in [7]. In other related work [4], the authors study commutativity in a duoidal setting which is related to the above mentioned notions of centre and centralisers. More recently, in [2], the authors established some additional results for the centre of a strong monad on a symmetric monoidal category and also studied the more general concept of central submonad. Inspired by these developments, in this paper we consider yet another notion of centre, this time for strong monads that are graded by a (partially ordered) monoid, which can be seen as an immediate generalization of the definition of centre proposed in [2].

Constructing the centre of a monad is a way to recover the commutativity of it by removing the effects that violate it. In practice, commutativity of a monad is an important property which means that the effect can occur deterministically inside operations without forcing the evaluation order (most compilers are allowed to evaluate operands in the order of their choice for optimization purposes). A Graded monad, however, can be seen as a form of statistical analysis that gives insight on the effects that can occur. We think that studying the interaction between these structures is important.

Before we present the technical results, we provide some computational intuition for the centre of a strong monad that may be useful to some readers.

Strong monads on a symmetric monoidal category can be used to represent sequencing of computational effects. Informally, suppose we are given an effectful computation $f\colon X_1 \to X_2$ acting on some variable $x_1\colon X_1$ and another effectful computation $g\colon Y_1 \to Y_2$ acting on a different variable $y_1\colon Y_1$. Since these two computations are effectful, the two ways of sequencing the computations

$$\text{do } x_2 \leftarrow f(x_1); y_2 \leftarrow g(y_1); h(x_2, y_2)$$

and

$$\text{do } y_2 \leftarrow g(y_1); x_2 \leftarrow f(x_1); h(x_2, y_2)$$

do *not* necessarily have the same computational result, even though the two computations are acting on different variables that have different types. When we interpret the above computational situation in the Kleisli category $\mathbf{C}_T$ of a strong monad $T\colon \mathbf{C} \to \mathbf{C}$, this potential difference is reflected by the fact that $\mathbf{C}_T$ has a *premonoidal* structure [9], rather than a monoidal one, and the premonoidal product is not bifunctorial (in general), unlike the monoidal one. When the effect under consideration is commutative and so the monad T is also *commutative* (not just strong), then the two ways of sequencing above have the same computational result and this is reflected in the Kleisli category $\mathbf{C}_T$ which has a monoidal structure (not just premonoidal) in this case. So, we may naturally arrive at the notion of *centre* of a strong monad by identifying all the central elements/effects, i.e. those that commute with all other elements/effects. This is the approach taken in [2] and in this paper we consider a more general setting.

More specifically, we consider a wider range of effects, namely those that can be described by pomonoid-graded strong monads, and then we propose a definition for the centre of such monads. The construction is analogous, but more general, compared to the one in [2]. In the last section, we investigate possible research perspectives which exploit our understanding of the preliminary study performed here. In particular, we acknowledge the fact that the centre of graded monads is seldom usable in practice as it is even more constrained than that of a monad. Finally, we open the discussion to relaxations in which we may eventually obtain more refined notions of centres.

2 Graded Monads

There are different ways of introducing graded monads in the literature [3,1,8]. They all share the same structure: using a monoid to index functors which, together, behave like a monad whose multiplication is graded by that of the monoid. It is then possible to extend the monoid with additional structures such as that of a semiring to represent additive monads, for example. We present the most common of these extensions, that of gradations by pomonoids (partially ordered monoids), whose order represents a degree of knowledge and which is essential for most applications that use statistical analysis [6].

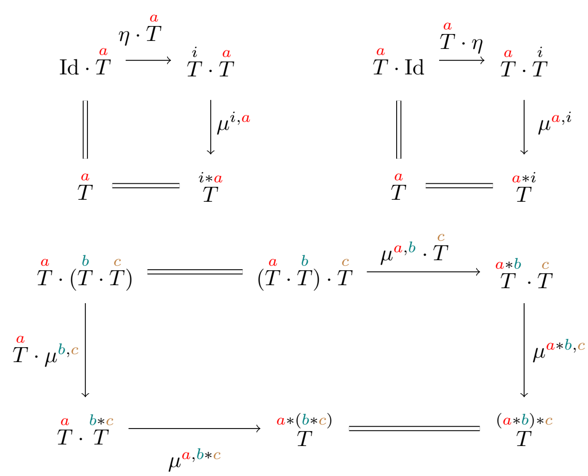
2.1 Graded Monads

Before we present the full definition of a pomonoid-graded monad, we start with a monoid-graded monad so that readers can hopefully acquire a better intuition for the more general notion that follows afterwards.

Definition 1 (Monoid Graded Monads). *Let $\mathcal{G} = (G, i, *)$ be a monoid. A $\mathcal{G}$-graded monad on a category $\mathbf{C}$ is given by the following data:*

- *for any $a \in G$, an endofunctor $\overset{a}{T} : \mathbf{C} \to \mathbf{C}$;*
- *a natural transformation $\eta : \mathrm{Id} \to \overset{i}{T}$;*
- *for any $a, b \in G$, a natural transformation $\mu^{a,b} : \overset{a}{T} \cdot \overset{b}{T} \to \overset{a*b}{T}$, where $(- \cdot -)$ indicates functor composition (we sometimes omit the gradations on μ for brevity);*

such that the following diagrams commute:



Remark 1. Every monad $T : \mathbf{C} \to \mathbf{C}$ can be seen as a $\mathcal{G}$-graded monad by setting $\mathcal{G}$ to be a singleton (i.e. trivial) monoid.

Next, we introduce pomonoids and the appropriate notion of morphism between them. This would then allow us to introduce a more flexible notion of gradation for our monads.

Definition 2 (Pomonoid [5]). *A* partially ordered monoid (pomonoid) *is a tuple $((G, \leq), i, *)$, where $(G, i, *)$ is a monoid and where $\leq$ is a partial order on G, such that the monoid operation is monotone with respect to the order in the following sense: for all $w, x, y, z \in G$, if $w \leq x$ and $y \leq z$, then $w * y \leq x * z$.*

Definition 3 (Morphism between Pomonids). *Let $\mathcal{G} = ((G, \leq), i, *)$ and $\mathcal{H} = ((H, \sqsubseteq), e, \circledast)$ be two pomonoids. A morphism between pomonoids is a function $\phi : G \to H$, such that:*

– $e \sqsubseteq \phi(i)$
– $\phi(x) \circledast \phi(y) \sqsubseteq \phi(x * y)$ *for all* $x, y \in G$.

Remark 2. A pomonoid $((G, \leq), i, *)$ can also be seen as a monoidal category $\mathbf{C}$ that is both skeletal and thin in the following way:

– objects of $\mathbf{C}$ are given by the elements $a \in G$;
– $\mathbf{C}$ has a unique morphism $a \to b$ iff $a \leq b$ in G;
– the tensor product is given by $a \otimes b \overset{\mathrm{def}}{=} a * b$;
– the tensor unit is given by i.

This correspondence may also be extended to cover morphisms between pomonoids: if $\phi \colon \mathcal{G} \to \mathcal{H}$ is a morphism of pomonoids, then ϕ may be identified with a lax monoidal functor between the (thin and skeletal) monoidal categories that represent $\mathcal{G}$ and $\mathcal{H}$.

Definition 4 (Pomonoid Graded Monad). *A* pomonoid-graded monad *on a category* $\mathbf{C}$ *is given by the following data:*

– *a pomonoid* $\mathcal{G} = ((G, \leq), i, *)$*, whose elements* $a \in G$ *we call* gradations,
– *for any* $a \in G$*, an endofunctor* $\overset{a}{T} : \mathbf{C} \to \mathbf{C}$*;*
– *a natural transformation* $\eta : \mathrm{Id} \to \overset{i}{T}$*;*
– *for any* $a, b \in G$*, a natural transformation* $\mu^{a,b} : \overset{a}{T} \cdot \overset{b}{T} \to \overset{a*b}{T}$
– *such that the following diagrams commute:*

$$\begin{array}{ccc} \mathrm{Id}\cdot\overset{a}{T} & \xrightarrow{\eta\cdot\overset{a}{T}} & \overset{i}{T}\cdot\overset{a}{T} \\ \Vert & & \downarrow{\scriptstyle \mu^{i,a}} \\ \overset{a}{T} & = & \overset{i*a}{T} \end{array} \qquad \begin{array}{ccc} \overset{a}{T}\cdot\mathrm{Id} & \xrightarrow{\overset{a}{T}\cdot\eta} & \overset{a}{T}\cdot\overset{i}{T} \\ \Vert & & \downarrow{\scriptstyle \mu^{a,i}} \\ \overset{a}{T} & = & \overset{a*i}{T} \end{array}$$

$$\begin{array}{ccccc} \overset{a}{T}\cdot(\overset{b}{T}\cdot\overset{c}{T}) & = & (\overset{a}{T}\cdot\overset{b}{T})\cdot\overset{c}{T} & \xrightarrow{\mu^{a,b}\cdot\overset{c}{T}} & \overset{a*b}{T}\cdot\overset{c}{T} \\ {\scriptstyle \overset{a}{T}\cdot\mu^{b,c}}\downarrow & & & & \downarrow{\scriptstyle \mu^{a*b,c}} \\ \overset{a}{T}\cdot\overset{b*c}{T} & \xrightarrow[\mu^{a,b*c}]{} & \overset{a*(b*c)}{T} & = & \overset{(a*b)*c}{T} \end{array}$$

– *for any* $a \leq a'$ *in* $\mathcal{G}$*, a natural transformation* $\overset{a\leq a'}{T} : \overset{a}{T} \to \overset{a'}{T}$*, such that:*

$$\overset{a\leq a}{T} = \iota \text{ (identity natural transformation)} \quad , \quad \overset{a'\leq a''}{T} \circ \overset{a\leq a'}{T} = \overset{a\leq a''}{T} \quad \textit{and}$$

$$\begin{array}{ccc} \overset{a}{T}\cdot\overset{b}{T} & \xrightarrow{\mu^{a,b}} & \overset{a*b}{T} \\ {\scriptstyle \overset{a\leq a'}{T}\cdot\overset{b\leq b'}{T}}\downarrow & & \downarrow{\scriptstyle \overset{a*b\leq a'*b'}{T}} \\ \overset{a'}{T}\cdot\overset{b'}{T} & \xrightarrow[\mu^{a',b'}]{} & \overset{a'*b'}{T} \end{array}$$

Remark 3. T can be seen as a bifunctor $\mathcal{G} \times \mathbf{C} \to \mathbf{C}$, with the last conditions being the functoriality and the naturality of μ with respect to $\mathcal{G}$. In fact, we can go even further: graded monads over $\mathcal{G}$ on a category $\mathbf{C}$ are exactly lax-monoidal functors (T, μ, η) from $(\mathcal{G}, *, i)$, seen as a thin monoidal category, to $([\mathbf{C}, \mathbf{C}], \cdot, \mathrm{id})$.

Example 1. Let T be a strong monad and Z its centre, then we can always construct a monad graded by $((\mathtt{Bool}, \mathtt{tt} \leq \mathtt{ff}), \mathtt{tt}, \wedge)$ defined by $\overset{\mathtt{tt}}{T} = Z$ and $\overset{\mathtt{ff}}{T} = T$ that keeps track whether we are in the centre or not. Notice that the morphism $\overset{\mathtt{tt}\leq\mathtt{ff}}{T} : Z \to T$ is the submonad monomorphism (inclusion). Such gradations can be used to infer whether an operation $\odot : A \times B \to C$ can be lifted into an effectful environment $\hat{\odot} : TA \times TB \to TC$ and still be evaluated freely by the compiler. Indeed, the graded version is of the form $\hat{\odot} : \overset{\mathtt{a}}{T}A \times \overset{\mathtt{b}}{T}B \to \overset{\mathtt{a}\wedge\mathtt{b}}{T}C$, the evaluation order have to be forced if $a = b = \mathtt{ff}$ but is commutative otherwise.

The notion of centre (and centrality) can be formulated for *strong* monads, as was previously argued in [2]. The corresponding definition for pomonoid-graded monads is presented next.

Definition 5 (Strong Pomonoid Graded Monad). *Let $\mathcal{G} = ((G, \leq), i, *)$ be a pomonoid. A* strong *$\mathcal{G}$-graded monad over a monoidal category $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ is a $\mathcal{G}$-graded monad (T, η, μ) equipped with a family of natural transformations $\overset{a}{\tau}_{X,Y} : X \otimes \overset{a}{T}Y \to \overset{a}{T}(X \otimes Y)$, indexed by elements $a \in \mathcal{G}$, called* graded strength*, such that for any element $b \in G$ and objects X, Y in $\mathbf{C}$, the following diagrams* *commute:*

$$
\begin{array}{ccc}
I \otimes \overset{a}{T}X & \xrightarrow{\tau_{I,X}} & \overset{a}{T}(I \otimes X) \\
& \searrow^{\lambda_{\overset{a}{T}X}} & \downarrow{\scriptstyle \overset{a}{T}\lambda_X} \\
& & \overset{a}{T}X
\end{array}
\qquad
\begin{array}{ccc}
X \otimes Y & \xrightarrow{X \otimes \eta_Y} & X \otimes \overset{i}{T}Y \\
& \searrow^{\eta_{X \otimes Y}} & \downarrow{\scriptstyle \tau_{X,Y}} \\
& & \overset{i}{T}(X \otimes Y)
\end{array}
$$

$$
\begin{array}{ccccc}
(W \otimes X) \otimes \overset{a}{T}Y & & \xrightarrow{\tau_{W \otimes X,Y}} & & \overset{a}{T}((W \otimes X) \otimes Y) \\
\downarrow{\scriptstyle \alpha_{W,X,\overset{a}{T}Y}} & & & & \downarrow{\scriptstyle \overset{a}{T}\alpha_{W,X,Y}} \\
W \otimes (X \otimes \overset{a}{T}Y) & \xrightarrow[W \otimes \tau_{X,Y}]{} & W \otimes \overset{a}{T}(X \otimes Y) & \xrightarrow[\tau_{W,X \otimes Y}]{} & \overset{a}{T}(W \otimes (X \otimes Y))
\end{array}
$$

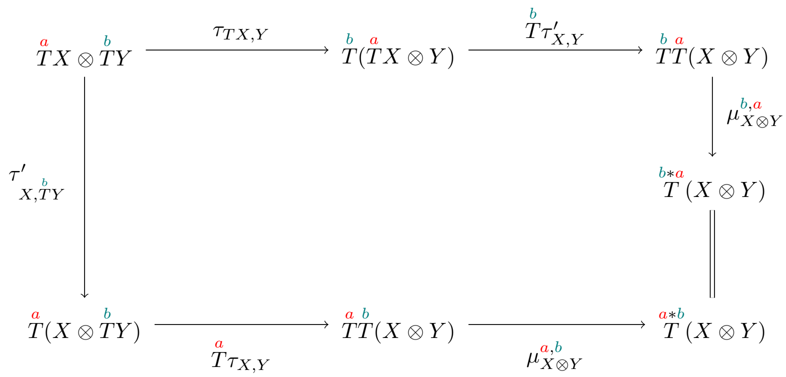
$$
\begin{array}{ccccc}
X \otimes (\overset{a}{T}\overset{b}{T}Y) & \xrightarrow{\tau_{X,TY}} & \overset{a}{T}(X \otimes \overset{b}{T}Y) & \xrightarrow{\overset{a}{T}\tau_{X,Y}} & \overset{a}{T}\overset{b}{T}(X \otimes Y) \\
\downarrow{\scriptstyle X \otimes \mu_Y^{a,b}} & & & & \downarrow{\scriptstyle \mu_{X \otimes Y}^{a,b}} \\
X \otimes \overset{a*b}{T}Y & & \xrightarrow[\tau_{X,Y}]{} & & \overset{a*b}{T}(X \otimes Y)
\end{array}
$$

where (for simplicity) we omitted the superscripts of τ*.*

If, moreover, we are given a *symmetric* monoidal category $(\mathbf{C}, \otimes, I, \gamma)$, then we can define the $\mathcal{G}$*-graded costrength* $\overset{a}{\tau'}_{X,Y} : \overset{a}{T}X \otimes Y \to \overset{a}{T}(X \otimes Y)$ by $\overset{a}{\tau'}_{X,Y} \overset{\text{def}}{=} \overset{a}{T}(\gamma_{Y,X}) \circ \overset{a}{\tau}_{Y,X} \circ \gamma_{\overset{a}{T}X,Y}$, where γ represents the symmetry. Note that this is completely analogous to how the costrength is defined without for monads that are not graded. We also often omit the gradations on τ' for convenience and note that it satisfies similar coherence conditions to that of the strength τ.

We can now introduce commutative graded monads whose definition is also analogous to the definition of a commutative monad that is not graded.

Definition 6 (Commutative Graded Monad). *Let* $\mathcal{G} = ((G, \leq), i, *)$ *be a* commutative *pomonoid. Let* (T, η, μ, τ) *be a strong* $\mathcal{G}$*-graded monad on a* symmetric *monoidal category* $(\mathbf{C}, \otimes, I, \gamma)$*. Then,* T *is said to be* commutative *if for* *any* $a, b \in G$*, and any objects* $X, Y \in \mathbf{C}$*, the following diagram commutes:*

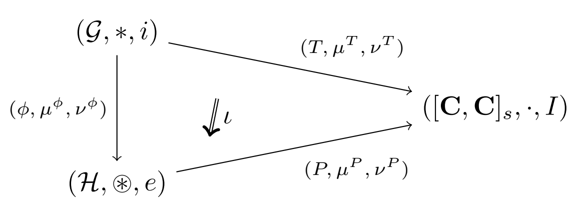


Remark 4. Note that in the above definition, the pomonoid is assumed to be commutative, so that $a * b = b * a$ which justifies the equality in the bottom right corner.

2.2 Morphisms between Strong Graded Monads

One of the issues that we have to address is how to formulate an appropriate definition of the morphisms between strong graded monads. This is important for our development as it underpins subsequent constructions that are relevant for the construction of the centre.

In order to visualize the definition of morphism between a strong $\mathcal{G}$-graded monad T and a $\mathcal{H}$-graded monad P over a category $\mathbf{C}$, remember that they can be seen as lax monoidal functors $T : \mathcal{G} \to [\mathbf{C}, \mathbf{C}]_s$ and $P : \mathcal{H} \to [\mathbf{C}, \mathbf{C}]_s$ targeting the same monoidal category, that of endofunctors $[\mathbf{C}, \mathbf{C}]_s$ over $\mathbf{C}$. Therefore, the morphisms between them can be identified with the morphisms of the lax-slice category over $[\mathbf{C}, \mathbf{C}]_s$ in `MonCat`.



Definition 7 (Morphism between Strong Graded Monads). *Let* $\mathcal{G} = ((G, \leq), i, *)$ *and* $\mathcal{H} = ((H, \sqsubseteq), e, \circledast)$ *be two pomonoids. Let* $(T, \eta^T, \mu^T, \tau^T)$ *be a* $\mathcal{G}$*-graded strong monad and let* $(P, \eta^P, \mu^P, \tau^P)$ *be an* $\mathcal{H}$*-graded strong monad over a symmetric monoidal category* $\mathbf{C} = (C, \otimes, I, \gamma)$*. A* morphism of strong graded monads *is given by the following data:*

- *a pomonoid-morphism* $\phi \colon \mathcal{G} \to \mathcal{H}$ *between the gradations;*

- *a family of natural transformations* $\overset{a}{\iota} : \overset{a}{T} \Rightarrow \overset{\phi a}{P}$ *indexed by elements of* $\mathcal{G}$[2],
- *such that for all* a, b *in* $\mathcal{G}$ *and* X, Y *in* $\mathbf{C}$, *the following diagrams commute:*

$$
\begin{CD}
X @>{\eta^P_X}>> \overset{e}{P}X \\
@V{\eta^T_X}VV @VV{\overset{e \sqsubseteq \phi(i)}{P}}V \\
\overset{i}{T}X @>>{\iota_X}> \overset{\phi i}{P}X
\end{CD}
\qquad\qquad
\begin{CD}
X \otimes \overset{a}{T}Y @>{X \otimes \iota_Y}>> X \otimes \overset{\phi a}{P}Y \\
@V{\tau^T_{X,Y}}VV @VV{\tau^P_{X,Y}}V \\
\overset{a}{T}(X \otimes Y) @>>{\iota_{X \otimes Y}}> \overset{\phi a}{P}(X \otimes Y)
\end{CD}
$$

$$
\begin{array}{ccccccc}
\overset{a}{T}\overset{b}{T}X & \xrightarrow{\ \iota\ } & \overset{\phi a}{P}\overset{b}{T}X & \xrightarrow{\overset{\phi a}{P}\iota_X} & \overset{\phi a}{P}\overset{\phi b}{P}X & \xrightarrow{\mu^P_X} & \overset{\phi a \circledast \phi b}{P}X \\
\Big\downarrow{\scriptstyle \mu^T_X} & & & & & & \Big\downarrow{\scriptstyle \overset{\phi a \circledast \phi b \sqsubseteq \phi(a * b)}{P}} \\
\overset{a * b}{T}X & & & \xrightarrow[\iota_X]{\hspace{6em}} & & & \overset{\phi(a * b)}{P}X
\end{array}
$$

The subtlety here is to explain the relations between $\overset{e}{P}$, $\overset{\phi i}{P}$, $\overset{\phi a \circledast \phi b}{P}$ and $\overset{\phi(a * b)}{P}$. This comes from the fact that the inequalities between the gradations induce natural transformations between the indexed endofunctors.

Remark 5. If we equip the pomonoids $\mathcal{G}$ and $\mathcal{H}$ with the discrete order, then the inequalities under consideration become the usual equalities for a monoid homomorphism, i.e. $\phi(i) = e$ and $\phi(a * b) = \phi a \circledast \phi b$. Then we can obviously identify $\overset{\phi i}{P}X = \overset{e}{P}X$ and $\overset{\phi(a * b)}{P}X = \overset{\phi a \circledast \phi b}{P}X$, so that we recover a notion of morphism of strong monoid-graded monads as a special case.

3 The Centre of a Strong Graded Monad

We can now outline our construction for the centre of a strong graded monad. Our construction is similar to the construction of the centre of a strong (not graded) monad in [2], but some of our proofs are established in a slightly different way. For example, some of the required proofs in [2] use results from the theory of premonoidal categories [9]. However, in our setting, this is not so straightforward, because in order to adapt the proofs, we would have to identify a suitable "premonoidal" theory for strong graded monads. Instead of doing this, we opt for a more direct approach that boils down to fairly large diagrammatic chasing in a few cases (see Appendix A). In particular, this gives us (as a special case) another proof for some of the established results in [2].

A basic intuition that we have for the centre of any algebraic structure is that it enjoys commutativity properties. The definition of a commutative graded monad (Definition 6) suggests that we should also take the centre of the gradations into account as well. This brings us to our next definition.

[2] We often omit the superscript of ι for convenience.

Definition 8 (Centre of a Pomonoid). *Given a pomonoid $\mathcal{G} = ((G, \leq), i, *)$, the* centre *of the pomonoid, written $Z(G)$, is given by the set*

$$Z(G) \stackrel{def}{=} \{z \in G |\ \forall b \in G,\ z * b = b * z\}.$$

It is easy to see that $\mathcal{Z}(\mathcal{G}) \stackrel{\text{def}}{=} ((Z(G), \leq), i, *)$ is a subpomonoid of $\mathcal{G}$ in the sense that the subset inclusion $\phi\colon Z(G) \subseteq G$ is a pomonoid-morphism.

Remark 6. The centre of a pomonoid, as given above, does not take the order into consideration. It will be interesting to identify a more general and flexible notion of centre that depends on the order and use that in future work.

Next we explain how to construct the centre of strong graded monads on **Set**, which we hope would help readers in understanding the more general construction that follows afterwards.

Definition 9 (Graded Centre in Set). *Let $\mathcal{Z}(\mathcal{G}) = ((Z(G), \leq), i, *)$ be a pomonoid which is the centre of the pomonoid $\mathcal{G} = ((G, \leq), i, *)$. Assume further that we are given a strong $\mathcal{G}$-graded monad (T, η, μ, τ) on* **Set**.

Given an arbitrary $z \in Z(G)$ and set X, we say that the $\mathcal{Z}(\mathcal{G})$-graded centre *of T at (z, X), written $\overset{z}{\mathcal{Z}}X$, is the set*

$$\overset{z}{\mathcal{Z}}X \stackrel{def}{=} \left\{ t \in \overset{z}{T}X \;\middle|\; \begin{array}{l} \forall b \in G, \forall Y \in \mathrm{Ob}(\mathbf{Set}), \forall s \in \overset{b}{T}Y, \\ \mu(\overset{z}{T}\tau'(\tau(t,s))) = \mu(\overset{b}{T}\tau(\tau'(t,s))) \end{array} \right\}.$$

We write $\overset{z}{\iota}_X : \overset{z}{\mathcal{Z}}X \subseteq \overset{z}{T}X$ for the indicated subset inclusion.

The main idea for this definition is to consider all of the monadic elements that satisfy the equation in Definition 6, for suitably fixed X and $z \in Z(G)$.

Notation 7 *For the rest of this section, we assume that* **C** *is a symmetric monoidal category; V, W, X, Y are objects in* **C***; we are given a pomonoid $\mathcal{G} = ((G, \leq), i, *)$ with centre $\mathcal{Z}(\mathcal{G}) = ((Z(G), \leq), i, *)$; we also have a $\mathcal{G}$-graded strong monad T; we write $z \in Z(G)$ for central gradations and more generally we write $b, c \in G$ for gradations of $\mathcal{G}$.*

To extend the definition of the centre from **Set** to other categories, we introduce graded central cones.

Definition 10 (Graded Central Cone). *Let X be an object of* **C** *and $z \in Z(G)$. A* graded central cone *of a $\mathcal{G}$-graded strong monad T at (z, X), is given by a pair (Z, ι) of an object Z and a morphism $\iota\colon Z \to \overset{z}{T}X$, such that for any* *object Y in $\mathbf{C}$ and any $b \in G$, the following diagram commutes:*

$$
\begin{array}{ccccc}
Z \otimes \overset{b}{T}Y & \xrightarrow{\iota \otimes \overset{b}{T}Y} & \overset{z}{T}X \otimes \overset{b}{T}Y & \xrightarrow{\tau'} & \overset{z}{T}(X \otimes \overset{b}{T}Y) \\
\downarrow{\scriptstyle \iota \otimes \overset{b}{T}Y} & & & & \downarrow{\scriptstyle \overset{z}{T}\tau} \\
\overset{z}{T}X \otimes \overset{b}{T}Y & & & & \overset{z}{T}\overset{b}{T}(X \otimes Y) \\
\downarrow{\scriptstyle \tau} & & & & \downarrow{\scriptstyle \mu^{z,b}_{X \otimes Y}} \\
 & & & & \overset{z*b}{T}(X \otimes Y) \\
 & & & & \| \\
\overset{b}{T}(\overset{z}{T}X \otimes Y) & \xrightarrow[\overset{b}{T}\tau']{} & \overset{b}{T}\overset{z}{T}(X \otimes Y) & \xrightarrow[\mu^{b,z}_{X \otimes Y}]{} & \overset{b*z}{T}(X \otimes Y)
\end{array}
$$

If (Z', ι') and (Z, ι) are two graded central cones of T at (z, X), a morphism of graded central cones *$\varphi : (Z', \iota') \to (Z, \iota)$ is a morphism $\varphi : Z' \to Z$, such that $\iota \circ \varphi = \iota'$. Graded central cones of T at (z, X) form a category and a* terminal graded central cone *of T at (z, X) is a terminal object in that category.*

Proposition 1. *If a terminal graded central cone for a $\mathcal{G}$-graded strong monad T at (z, X) exists, then it is unique up to a unique isomorphism of graded central cones. Also, if (Z, ι) is a terminal graded central cone, then ι is a monomorphism.*

Proof. Straightforward, essentially the same as in [2].

In particular, Definition 9 gives a terminal graded central cone for the special case of graded monads over **Set**.

Definition 11 (Centralisable Graded Monad). *We say that the $\mathcal{G}$-graded monad T over $\mathbf{C}$ is* centralisable *if, for any object X in $\mathbf{C}$, for any element z in $\mathcal{Z}(\mathcal{G})$, a terminal graded central cone of T at (z, X) exists. In this situation, we write $(\overset{z}{\mathcal{Z}}X, \overset{z}{\iota}_X)$ for the terminal graded central cone of T at (z, X).*

For a centralisable $\mathcal{G}$-graded monad T, the next theorem shows that its terminal graded central cones induce a commutative $\mathcal{Z}(\mathcal{G})$-graded submonad $\mathcal{Z}$ of T, which we call the *centre* of T.

Theorem 1 (Centre). *If the $\mathcal{G}$-graded monad T is centralisable, then the assignment $\mathcal{Z}(-)$ extends to a commutative $Z(\mathcal{G})$-graded monad $(\mathcal{Z}, \eta^{\mathcal{Z}}, \mu^{\mathcal{Z}}, \tau^{\mathcal{Z}})$ on $\mathbf{C}$, called the* centre *of T. Moreover, $\mathcal{Z}$ is a commutative $\mathcal{Z}(\mathcal{G})$-graded submonad of T and the family of morphisms $\overset{z}{\iota}_X : \overset{z}{\mathcal{Z}}X \to \overset{z}{T}X$, determine a monomorphism of strong graded monads $\mathcal{Z} \to T$ in the sense of Definition 7.*

4 Examples of Centres of Strong Graded Monads

In this section we provide more examples of strong graded monads that admit centres. We begin with a concrete example in **Set** and we discuss the role of the gradations.

Example 2. The multi-error graded writer monad is a writer monad which stops the computation whenever it encounters an error. For the gradation, consider a monoid $\mathcal{G} = (\{t, e, w_1 ... w_n\}, t, *)$ of three kinds of possible outcomes: t indicates the result is not an error nor warning, w_i are arbitrary elements in the set of warnings, and e is an error. The neutral element is t.

The monoid composition is defined as follows:

$*$	t	e	w_a	w_b
t	t	e	w_a	w_b
e	e	e	e	e
w_a	w_a	e	w_a	w_b
w_b	w_b	e	w_a	w_b

The rules $w_a * w_b = w_b$ and $w_b * w_a = w_a$ indicate that we only keep track of the first warning encountered. Thus the centre of $\mathcal{G}$ is $\mathcal{Z}(\mathcal{G}) = (\{t, e\}, t, *)$.

Now we can define the multi-error writer monad as the writer monad T graded by $\mathcal{G}$, corresponding to the different annotations of outcomes:

$$\overset{e}{T}X \overset{\text{def}}{=} 1, \quad \overset{t}{T}X \overset{\text{def}}{=} X, \quad \overset{w_a}{T}X \overset{\text{def}}{=} X \times \{a\} \quad \overset{w_b}{T}X \overset{\text{def}}{=} X \times \{b\}.$$

The unit is given by $\eta(x) = x \in \overset{t}{T}$ and the multiplication by

$$\mu_{w_i,w_j}((x,i),j) \overset{\text{def}}{=} i, \quad \mu_{t,w_i}(x,i) \overset{\text{def}}{=} \mu_{w_i,t}(x,i) \overset{\text{def}}{=} (x,i), \quad \mu_{t,t}(x) \overset{\text{def}}{=} x.$$

Its centre is a commutative $(\{t, e\}, t, *)$-graded submonad. It consists of two endofunctors that are given by $\overset{e}{T}X = 1$ and $\overset{t}{T}X = X$.

The category **Set** is not the only category for which we can systematically construct centers of strong graded monads. Just like in [2], there are many such categories. Next, we provide such an example.

Example 3. Let T be a strong $\mathcal{G}$-graded monad on **Top**, the category of topological spaces and continuous maps between them. The strength is considered with respect to the Cartesian structure of **Top**. Then T is centralisable with terminal central cones given by

$$\overset{z}{\mathcal{Z}}X \overset{\text{def}}{=} \left\{ t \in \overset{z}{T}X \;\middle|\; \begin{array}{c} \forall b \in G, \forall Y \in \mathrm{Ob}(\mathbf{Top}), \forall s \in \overset{b}{T}Y, \\ \mu(\overset{z}{T}\tau'(\tau(t,s))) = \mu(\overset{b}{T}\tau(\tau'(t,s))) \end{array} \right\},$$

where, as usual, $z \in \mathcal{Z}(\mathcal{G})$ and the topology on $\overset{z}{\mathcal{Z}}X$ is the subspace topology inherited from the inclusion $\overset{z}{\mathcal{Z}}X \subseteq \overset{z}{T}X$.

The next example should be no surprise.

Example 4. Every commutative graded monad is naturally isomorphic to its centre.

It was shown in [2] that not every strong monad is centralisable. Since strong graded monads are more general, then it should be clear that not every strong graded monad has a centre. However, just like in [2], the notion of centre for strong graded monads is ubiquitous and we do not know of any natural counter-examples (i.e. counter-examples which were not constructed for this sole purpose as in [2]).

5 Central Graded Submonads

We can introduce *central graded submonads* of a strong graded monad in analogy to the construction in [2].

Theorem 2 (Centrality). *Let $\mathcal{Z}(\mathcal{G}) = (Z(G), i, *)$ be the centre of a pomonoid $\mathcal{G} = (G, i, *)$. Let $\mathbf{C}$ be a symmetric monoidal category and T a strong $\mathcal{G}$-graded monad on it. Assume that T has a centre $\mathcal{Z}$ that is graded by $\mathcal{Z}(\mathcal{G})$. Let $\mathcal{S}$ be a strong $\mathcal{Z}(\mathcal{G})$-graded submonad of T with $\iota : \mathcal{S} \to T$ the strong submonad monomorphism. The following are equivalent:*

1) *For any element z in $\mathcal{Z}(\mathcal{G})$, any object X of $\mathbf{C}$, we have that $(\overset{z}{\mathcal{S}}X, \overset{z}{\iota}_X)$ is a graded central cone for T at (z, X);*
2) *$\mathcal{S}$ is a commutative graded submonad of the centre of T with submonad morphism $\iota^{\mathcal{S}} \colon \mathcal{S} \to \mathcal{Z}$ such that $\iota \colon \mathcal{S} \to T$ factorises through $\iota^{\mathcal{S}}$.*

Proof. $(1 \Rightarrow 2)$. Commutativity follows with the same arguments as in Diagram 3 (Appendix A) by replacing $\mathcal{Z}$ with $\mathcal{S}$. Each $\overset{z}{\iota}_X \colon \overset{z}{\mathcal{S}}X \to \overset{z}{T}X$ factorizes through the terminal central cone $\iota^{\mathcal{Z}}_X$ (by definition). These factorisations determine the submonad morphism $\mathcal{S} \to \mathcal{Z}$.

$(2 \Rightarrow 1)$: This follows by Lemma 2 in Appendix A.

Definition 12 (Central Graded Submonad). *Given a strong graded submonad $\mathcal{S}$ of T, we say that $\mathcal{S}$ is a* central graded submonad *of T if it satisfies one of the two equivalent conditions from Theorem 2.*

Every central graded submonad is commutative and Theorem 2 shows that the centre (whenever it exists) can be seen as the largest central graded submonad of T.

6 Relaxations

It is unfortunate that the definition of centre, for graded monads, is not able to make use of the order of the pomonoid. When we started this project, we were hopping for the inclusion $\mathcal{Z}(\mathcal{G}, *) \subseteq \mathcal{G}$ to use a multiplication $(\circledast) \neq (*)$ which seems reasonable since the inclusion only requires that $a \circledast b \leq a * b$. Such a result would mean that the constructed centre would be able not only to restrict the graded monad to commutative elements, but also to preserve some non-commutative ones by approximating their interaction with others. The centre, however, is arguably a notion that is too universal and which we think prevents us from constructing such a non-free operation $(\circledast)$.

In this section, we explore perspectives that may allow us to overcome this issue and that make use of the order. Intuitively, one of the issues that we faced, was the absence of structure in the pomonoid other than the multiplication and the order. As a result of this, it is difficult, if not impossible, to canonically construct the new approximated multiplication. Therefore, we believe that we have to use more structure than that of a pomonoid. Towards this end, we can see at least three natural extensions:

- A natural extension consists in assuming that the commutative $(\circledast)$ operation is given in the pomonoid but not for the graded monad. We can then try to pull it over the gradation using the central construction.
- We could consider the order to be much richer, in particular, we could require the existence of a quantal, which is, intuitively, a pomonoid over a complete lattice. Then we could try to construct limits over all possible commutations of elements, generalizing the idea of shuffle. However, this is not possible to do in an arbitrary quantal, not even in every quantal over free complete lattices because this does not preserve the associativity of the new commutative operation.
- The last direction is relaxing the notion of commutativity itself, hence relaxing the constraints over the centre.

6.1 Bimonoids

Definition 13 (A Bimonoid). *A* bimonoid *is a pomonoid $(G, \leq, i, *)$ with an additional symmetric monoidal structure $(j, \circledast)$ such that*

$$a * b \leq a \circledast b \tag{1}$$

We use δ to refer to the above inequality.

Remark 8. The name bimonoid is used for many structures and the above definition is non-standard.

Property 1. Let $\mathcal{G} = (G, \leq, i, *, j, \circledast)$ be a bimonoid. An ordered $(\mathcal{G}, \leq, i, *)$-graded monad T on a monoidal category $\mathbf{C}$ is also an ordered $(\mathcal{G}, \leq, i, \circledast)$-graded monad with the same structure except for the multiplication $\mu^{\circledast}_{a,b,X} = \mu^* \quad \bar{T}^{a*b \leq a \circledast b}$.

Definition 14 (Bimonoidal Graded Centre in Set). *Let T be a strong $\mathcal{G}$-graded monad on* **Set***, so that $\mathcal{G}$ also has a bimonoidal operation $(\circledast)$.*

For an arbitrary $a \in \mathcal{G}$*, we say that the* $\mathcal{G}$*-graded* centre *of* T *relative to* $(\circledast)$ *at* (a, X)*, written* $\overset{a}{\mathcal{Z}}_{\circledast} X$*, is the set*

$$\overset{a}{\mathcal{Z}}_{\circledast} X \overset{def}{=} \left\{ t \in \overset{a}{T} X \;\middle|\; \begin{array}{c} \forall b \in G, \forall Y \in \mathrm{Ob}(\mathbf{Set}), \forall s \in \overset{a}{T} Y, \\ \overset{a*b \leq a \circledast b}{T} (\mu(\overset{a}{T} \tau'(\tau(t, s)))) = \overset{b*a \leq a \circledast b}{T} (\mu(\overset{b}{T} \tau(\tau'(t, s)))) \end{array} \right\}.$$

We write $\overset{a}{\iota}_X : \overset{a}{\mathcal{Z}}_{\circledast} X \subseteq \overset{a}{T} X$ *for the indicated subset inclusion.*

The $\mathcal{G}$-graded centre of T relative to $(\circledast)$ is then commutative on its $(\mathcal{G}, \circledast)$ gradation but not on its $(\mathcal{G}, *)$ gradation. Since the first gradation is an approximation of the second one, it is suitable for interpreting programs where we sometimes know of the evaluation order (e.g. composition) and where we sometimes do not (e.g. binary operators).

Example 5. A simple way to construct an example is to consider a monad graded by a pomonoid $\mathcal{G}$ with an absorbing top element $\top$. In this case we can define $a \circledast b$ to be $a * b$ when $a, b \in Z(\mathcal{G})$ and $a \circledast b = \top$ otherwise. This way, the restriction to the $\top$ grade is also a monad, and we work at the same time with the centre of the graded monad when using gradations $a \in Z(\mathcal{G})$, the centre of the $\overset{\top}{T}$ when we can't compute the grade, and still have access to the previous grades when not using $\mu^{\circledast}$.

This new notion of centre, which can be generalized to other categories, offers a sightly richer structure since the gradation is not $\mathcal{Z}(G)$ but $(G, \circledast)$. But it requires knowledge of $\circledast$ and many of the limitations of the centre still persist.

6.2 Quantal

The inspiration for our next idea is as follows: if we had access to sups in the pomonoid, we could try to construct $\circledast$.

A quantal is a monoid in the category of complete lattices and sup-preserving functions. Notice that the tensor product that we have to use on complete lattices is the right adjoint of the arrow $[A \Rightarrow B]$ of sup-preserving functions ordered pointwise. However, this is not an object that is easy to use. Fortunately, on complete lattices freely generated from a poset (the free functor is associating a poset to the complete lattice of its initial segments), it corresponds to the usual tensor on the underlying poset. We are thus focusing on the quantals obtained as the free completions of pomonoids.

Under some reasonable conditions (e.g. $\mathcal{G}$ has all κ-colimits, and $\overset{a}{T}$ are κ-ary functors) a monad graded by a pomonoid $\mathcal{G}$ seems to be definable as a monad graded by the quantal $\mathcal{I}(\mathcal{G})$ by computing the left Kan extension of $T : \mathcal{G} \to [\mathbf{C}, \mathbf{C}]$ along the inclusion $\mathcal{G} \to \mathcal{I}(\mathcal{G})$ in `MonCat`.[4]

[4] The difficulty here is that doing this in `MonCat` is much trickier than doing it in `Cat`.

Thus finding a commutative over-approximant $\circledast$ of $(*)$ in $\mathcal{I}(\mathcal{G})$ will permit, via the above construction, to obtain a commutative graded monad. The question is how to obtain $\circledast$.

A natural candidate is $a \circledast b \stackrel{\text{def}}{=} (a * b) \vee (b * a)$, but, unfortunately, it is generally not an associative operation. A more reasonable approach consists in computing the shuffle operation, or rather a generalization of it. In this case most examples generate an associative operation, but not all. A free construction is yet to be discovered for a good over-approximation.

6.3 Duoids and Concurrent Monads

In order to go in this last direction, we first need to investigate how to relax the notion of commutativity. Indeed, there is no way to change the monadic composition along this new operation over grades, but we can "approximate" the composition along this operation.

Commutativity, in its standard form, is difficult to relax naturally. However, one can present commutativity of a monad through its monoidality.

Lemma 1. *A monad T in a monoidal category $(\mathbf{C}, \otimes, e)$ is commutative iff there exists a natural transformation $m_{X,Y} : TX \otimes TY \to T(X \otimes Y)$ such that :*

$$\begin{array}{ccccc} TTX \otimes TTY & \xrightarrow{m} & T(TX \otimes TY) & \xrightarrow{Tm} & TT(X \otimes Y) \\ \downarrow{\scriptstyle \mu \otimes \mu} & & & & \downarrow{\scriptstyle \mu} \\ TX \otimes TY & & \xrightarrow{\quad m \quad} & & T(X \otimes Y) \end{array}$$

and

$$(\eta \otimes \eta); m = \eta \quad (m \otimes \mathrm{id}); m; T\alpha = \alpha(\mathrm{id} \otimes m); m$$

$$(\eta \otimes \mathrm{id}); m; T\lambda = \lambda \quad (\mathrm{id} \otimes \eta); m; T\rho = \rho$$

This definition can be relaxed by orienting the main diagram.

Definition 15 (Lax Commutative). *A monad T in an order-enriched monoidal category $(\mathbf{C}, \otimes, e)$ is said to be lax commutative iff there exists a natural transformation $m_{X,Y} : TX \otimes TY \to T(X \otimes Y)$ such that :*

$$\begin{array}{ccccc} TTX \otimes TTY & \xrightarrow{m} & T(TX \otimes TY) & \xrightarrow{Tm} & TT(X \otimes Y) \\ \downarrow{\scriptstyle \mu \otimes \mu} & & \Downarrow & & \downarrow{\scriptstyle \mu} \\ TX \otimes TY & & \xrightarrow{\quad m \quad} & & T(X \otimes Y) \end{array}$$

and

$$(\eta \otimes \eta); m = \eta \quad (m \otimes \mathrm{id}); m; T\alpha = \alpha(\mathrm{id} \otimes m); m$$

$$(\eta \otimes \mathrm{id}); m; T\lambda = \lambda \quad (\mathrm{id} \otimes \eta); m; T\rho = \rho$$

Such a definition means that the monad is not commutative, but can be over-approximated as such. In terms of proper programs, it means that the monad is not commutative but it has been extended with a kind of non-determinism so that one can approximate the operator effects by considering all their eventual behaviors (left-right, right-left, but also interleaving or parallelism).

This concept is closely related with that of a concurrent monad, a recent concept which is basically a lax-commutative monad with an additional relaxation on the unit. In this work, we do not dwell on concurrent monads, no more than we indulge in bicategories, but we do use the algebraic object of which the former is a categorification: the duoids.

Definition 16 (Duoid). *A duoid is a pomonoid $(G, \leq, i, *)$ with an additional symmetric monoidal structure $(j, \|)$ such that*

$$(a \parallel c) * (b \parallel d) \leq (a * b) \parallel (c * d) \tag{2}$$

We call δ the first inequality.

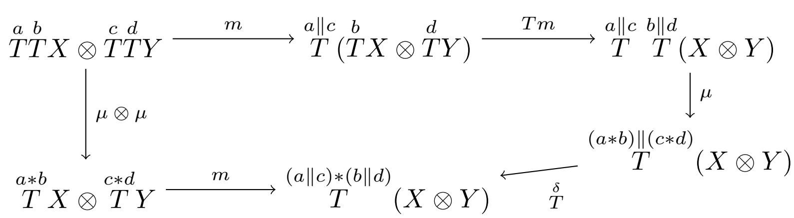
One can see this as a poset $(G, \leq)$ with two multiplicative operations $(*)$ and $(\|)$ making it a pomonoid in two different ways. The first half is non-commutative and represent the sequential composition, the second half is commutative and represents the parallel composition. The parallel composition can be seen as much less precise than the sequential composition. We can deduce from (2) that $a * b \leq a \parallel b$, i.e. $(\|)$ is approximating $(*)$. Therefore, it can play the rôle of $(\circledast)$ in the introduction.

Remark 9. The name duoid is used for many structures. Some non-equivalent definitions may or may not use another unit $i \leq j$ for the operation $(\|)$, they may or may not require symmetry/braiding of $(\|)$ and/or $(*)$, they also can be strict/weak/colax. It is important for us to use exactly the above definition.

Definition 17 (Duoidal Gradation). *Let $\mathcal{G} : (G, \leq, i, *, j, \|)$ be a duoid. A duoidal $\mathcal{G}$-graded monad on a monoidal category $\mathbf{C}$ is given by the following data:*

- *an ordered $(G, \leq, i, *)$-graded monad T*
- *a transformation $m_{a,b,X,Y} : \overset{a}{T}X \otimes \overset{b}{T}Y \to \overset{a\|b}{T}(X \otimes Y)$ natural in a, b, X and Y.*

such that the following diagrams commute:



as well as the other monoidal diagrams:

$$(\eta \otimes \eta); m = \eta \quad (m \otimes \mathrm{id}); m; T\alpha = \alpha(\mathrm{id} \otimes m); m$$

$$(\eta \otimes \mathrm{id}); m; T\lambda = \lambda \quad (\mathrm{id} \otimes \eta); m; T\rho = \rho$$

Example 6. Let Σ be an alphabet, then $\mathcal{P}(\Sigma^*)$ is a duoid with the concatenation and shuffle operations

$$L * L' \stackrel{\text{def}}{=} \{ww' \mid w \in L, w' \in L'\} \quad i = \{\epsilon\}$$

$$L \parallel L' \stackrel{\text{def}}{=} \{w_1 w_1' ... w_n w_n' \mid w_1 ... w_n \in L, w_1' ... w_n' \in L'\} \quad j = \epsilon$$

This duoid is grading the corresponding writer monad:

$$\overset{L}{T} X \stackrel{\text{def}}{=} \{(x, L') \mid x \in X, L' \subseteq L\}) \quad \overset{L}{T} f(x, L') \stackrel{\text{def}}{=} (f(x), L')$$

$$\overset{L \subseteq L'}{T} X \texttt{ is the inclusion of sets}$$

$$\eta(x) = (x, i) \qquad \mu((x, L), L') = (x, L * L')$$

$$m((x, L), (x', L')) = ((x, x'), L \parallel L')$$

Importing the construction inspired from the centre in such a duoidal framework may result in some more refined results. But it is still unclear (to us) how the notion of central cone can be remodeled to fit the monoidality diagram and the notion of lax commutativity.

Conclusion and Future Work

In this work, we showed how to construct the centre of graded monads and how to formulate central graded submonads, where the gradation is given by pomonoids and where the monads are defined over symmetric monoidal categories. We also introduce the lax commutativity on graded monads by choosing duoids as gradations, to make use of different pomonoid structures.

As part of future work, it will be interesting to see if there are other ways to take the order of a pomonoid into account when constructing the centre and central cones. Furthermore, we have not provided any suitable universal conditions (similar to the ones in [2]) that characterise the centre of a graded monad, so this is another open problem.

Acknowledgments

We thank Louis Lemonnier, Dylan McDermott and Tarmo Uustalu for discussions related to the paper. QL gratefully acknowledges financial support from LIPN which funded his internship during which a lot of the work was done.

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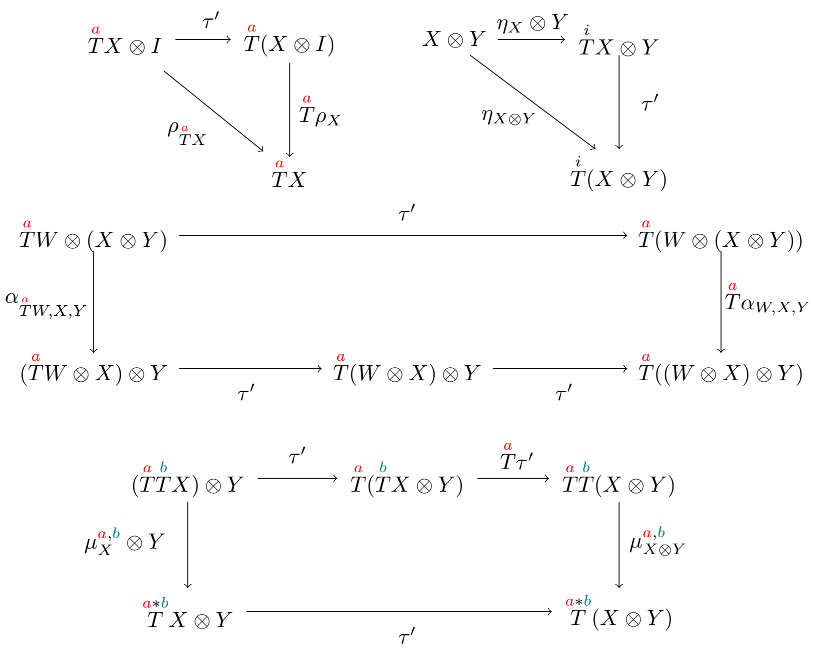
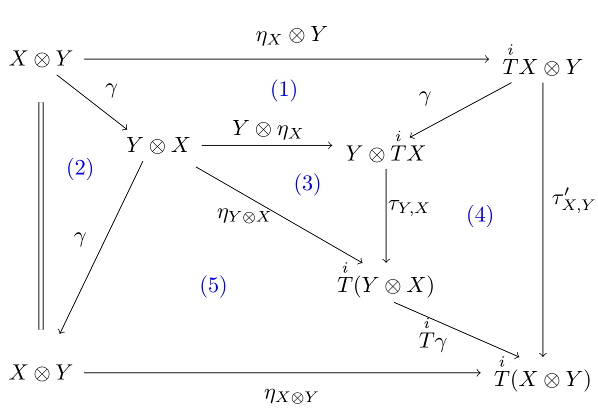
A Coherence Properties of costrength

Proposition 2 (Coherence Properties of costrength). *For all elements* z, b *in* G*,* X, Y *in* $\mathbf{C}$*, the following diagrams commute:*



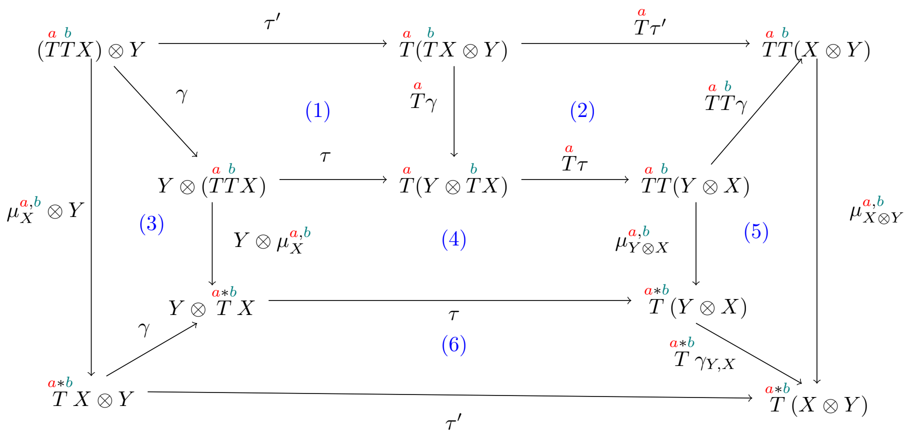
Proof. Only η and μ part of this proposition is going to be used in the proofs later, hence we only give proofs on those two parts, the rest could be proved similarly.

The proof of η :



(1) γ is natural; (2) $\gamma_{X,Y};\gamma_{Y,X} = id$; (3) definition of τ; (4) definition of τ'; and (5) η and γ are natural.

μ :

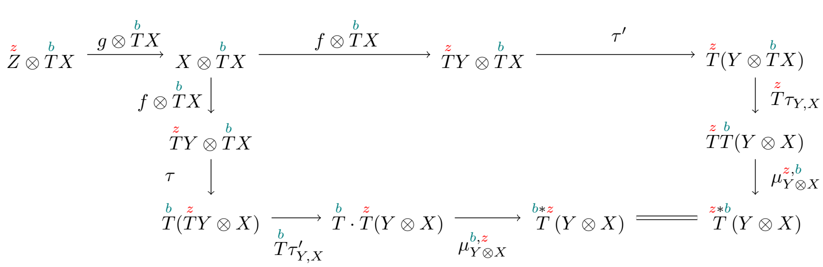


(1) fact that $\overset{a}{T}\gamma_{TX,Y} = \overset{a}{T}\gamma^{-1}_{Y,TX}$, and definition of τ'; (2) $\overset{a}{T}$ is a functor and definition of τ'; (3) γ is natural; (4) definition of strength; (5) $\mu^{a,b}_{X\otimes Y}$ and γ are natural and (6) definition of τ'.

B Lemmas for proof of Theorem 1

Lemma 2. *If $(X, f : X \to \overset{z}{T}Y)$ is a graded central cone of T at (z, Y). Then for any $g : Z \to X$ in* **C***, it follows that $(Z, f \circ g)$ is a graded central cone of T at (z, Y).*

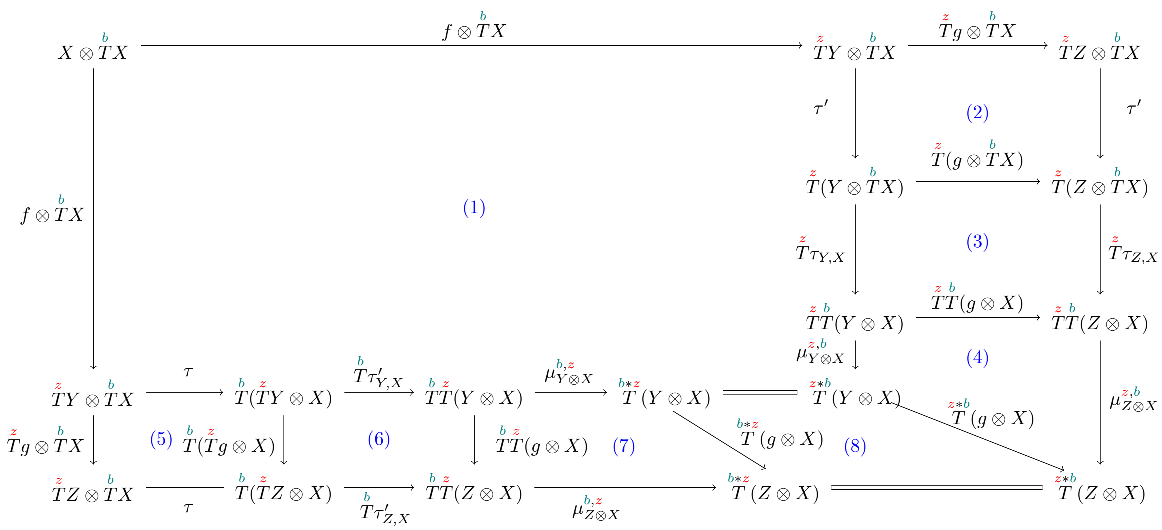
Proof. This is obtained by precomposing the definition of graded central cone by $g \otimes \mathrm{id}$. For all $b \in G$ and X in **C**,



commutes directly from the definition of graded central cone for f.

Lemma 3. *If* $(X, f : X \to \overset{z}{T}Y)$ *is a graded central cone of* T *at* (z, Y)*. Then for any* $g : Y \to Z$ *in* $\mathbf{C}$ *at* z*, it follows that* $(X, \overset{z}{T}g \circ f)$ *is a graded central cone of* T *at* Z *at* z*.*

Proof. The naturality of τ and μ allow us to push the application of g to the last postcomposition, in order to use the central property of f. In more details, for all $b \in G$ and X in $\mathbf{C}$, the following diagram:



commutes, because: (1) f is a graded central cone, (2) τ' is natural, (3) τ is natural, (4) μ is natural (5) τ is natural, (6) τ' is natural, (7) μ is natural, (8) T is a functor.

Lemma 4. *If* (Z, ι) *is a terminal (graded) central cone of* T *at* X*, then* ι *is a monomorphism.*

Proof. Let us consider $f, g : Y \to Z$ such that $\iota \circ f = \iota \circ g$; this family of morphism is a graded central cone at X (Lemma 2), and since (Z, ι) is a terminal graded central cone, it factors uniquely through ι. Thus $f = g$ and therefore ι is monic.

Lemma 5. *For* $A := (W \otimes TX) \otimes Y$

$$\tau'_{W \otimes X, Y} \circ \tau_{W,X} \otimes Y \circ A = T\alpha^{-1}_{W,X,Y} \circ \tau_{W, X \otimes Y} \circ W \otimes \tau'_{X,Y} \circ \alpha_{W,TX,Y} \circ A$$

Proof. Left $= T(W \otimes X) \otimes Y =$ Right.

C Proof of Theorem 1

Proof (Proof of Theorem 1).

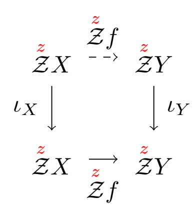
This proof has 3 parts.

First we prove that $\mathcal{Z}$ is a functor; then we give its graded monad structure and prove it, by showing all of its morphisms exist and are unique, and also natural; last we show that it's a strong and commutative graded monad.

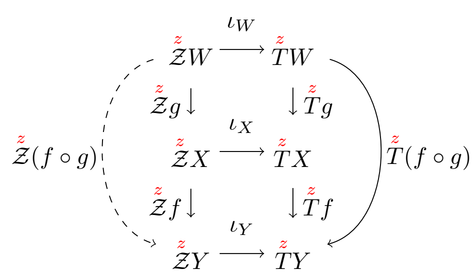
First part:

The first part is following same proof strategy as that of the Theorem in [2], but on graded monads, which are lax monoidal functors between gradation and endofunctor category of $\mathbf{C}$.

Recall that $\overset{z}{\mathcal{Z}}$ maps every object X to its terminal central cone at (z, X). Let $f : X \to Y$ be a morphism. $\overset{z}{T} f \circ \overset{z}{\iota}_X : \overset{z}{\mathcal{Z}} X \to \overset{z}{T} Y$ is a central cone according to Lemma 3. Therefore, by proposition 1, we can define $\overset{z}{\mathcal{Z}} f$ as the unique map such that the following diagram commutes:



It follows directly that $\overset{z}{\mathcal{Z}}$ maps the identity to the identity, and that $\overset{z}{\iota}$ is natural. $\overset{z}{\mathcal{Z}}$ also preserves composition, which follows by the commutative diagram below:



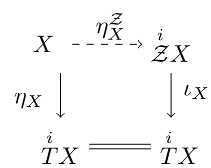
This proves that $\overset{z}{\mathcal{Z}}$ is a functor.

Second part.

Then we describe its graded monad structure and prove all its morphisms exist and are unique.

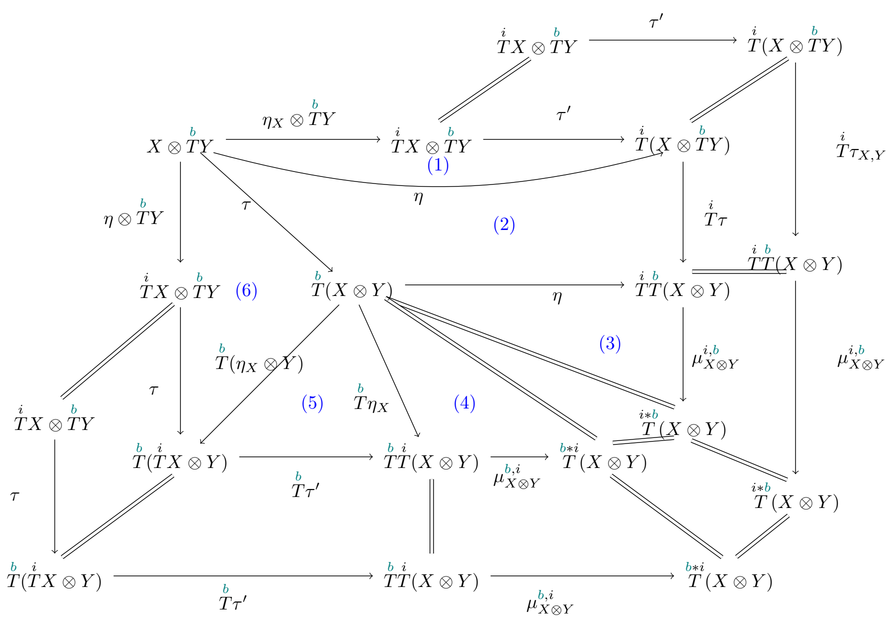
This part we give different proof strategy from that of [2], which bypasses the Kleisli constructure and independent of premonoidal center.

Definition of the monadic unit $\eta_X^{\mathcal{Z}}$ (the inequality arrow with respect to Definition 7 becomes equal because now it's inclusion):



By Definition 10, the universal property of terminal graded central cone indicates that, for any graded central cone at (z, X), there exists a unique morphism of graded central cones to $\overset{z}{\mathcal{Z}}$, hence we need to prove that all other arrows in this definition form graded central cones.

What's left is that η_X forms a central cone, it is proved by the following diagram:



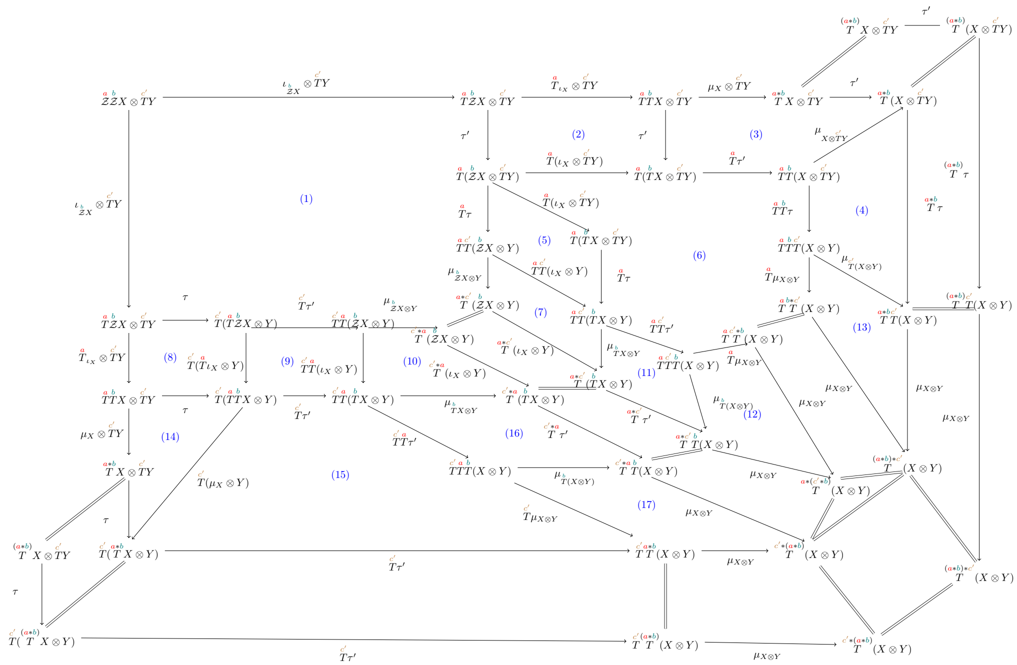
(1) η property of graded co-strength; (2) η and τ are natural; (3),(4) definition of graded monad; (5) T is a functor with η property of graded co-strength; (6) τ is natural and rest are equalities.

Next, definition of multiplication $\mu^{\mathcal{Z}}_X$ (the inequality arrow with respect to Definition 7 becomes equal because now it's inclusion):

$$\begin{array}{ccccccc}
\overset{a}{\mathcal{Z}}\overset{b}{\mathcal{Z}}X & & \xrightarrow{\quad\mu^{\mathcal{Z}}_X\quad} & & & & \overset{a*b}{\mathcal{Z}}X \\
\downarrow{\scriptstyle \iota_{\overset{b}{\mathcal{Z}}X}} & & & & & & \downarrow{\scriptstyle \iota_X} \\
\overset{a}{T}\overset{b}{\mathcal{Z}}X & \xrightarrow[\overset{a}{T}\iota_X]{} & \overset{a}{T}\overset{b}{T}X & \xrightarrow[\mu^T_X]{} & \overset{a*b}{T}X & = & \overset{a*b}{T}X
\end{array}$$

By Definition 10, the universal property of terminal graded central cone indicates that, for any graded central cone at (z, X), there exists a unique morphism of graded central cones to $\overset{z}{\mathcal{Z}}$, hence we need to prove that all other arrows in this definition form graded central cones.

What's left is that $\mu^T_X \circ \overset{a}{T}\iota_X \circ \iota_{\overset{b}{\mathcal{Z}}X}$ forms a central cone, it is proved by the following diagram:



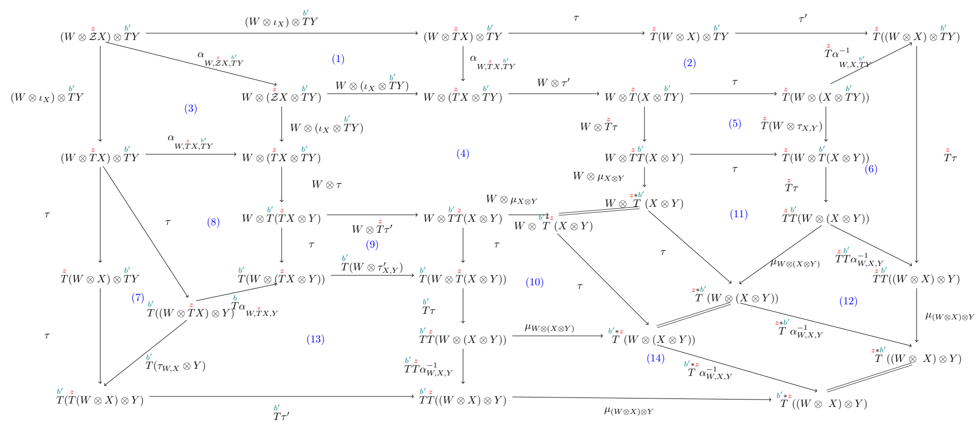
(1) $\iota_{\mathcal{Z}X}$ is central; (2) τ' and ι_X are natural; (3) proposition 2, μ property of graded co-strength (take normal monad as special case of graded monad); (4) μ and τ are natural; (5) T is a functor and τ, ι_X are natural; (6) T is a functor and ι_X is central; (7) μ and ι_X are natural; (8) τ and ι_X are natural; (9) T is a functor and τ', ι_X are natural; (10) μ and ι_X are natural; (11) μ and τ' are natural; (12),(13) and (17) definition of monad; (14) μ and τ are natural; (15) T is a functor and proposition 2, μ property of graded co-strength; (16) μ and τ' are natural and rest are equalities.

Last, definition of the strength $\tau^{\mathcal{Z}}_{W,X}$:

$$\begin{array}{ccc} W \otimes \overset{z}{\mathcal{Z}}X & \overset{\tau^{\mathcal{Z}}_{W,X}}{\dashrightarrow} & \overset{z}{\mathcal{Z}}(W \otimes X) \\ {\scriptstyle W \otimes \iota_X}\downarrow & & \downarrow{\scriptstyle \iota_{W \otimes X}} \\ W \otimes \overset{z}{T}X & \underset{\tau_{W,X}}{\longrightarrow} & \overset{z}{T}(W \otimes X) \end{array}$$

By Definition 10, the universal property of terminal graded central cone indicates that, for any graded central cone at (z, X), there exists a unique morphism of graded central cones to $\overset{z}{\mathcal{Z}}$, hence we need to prove that all other arrows in this definition form graded central cones.

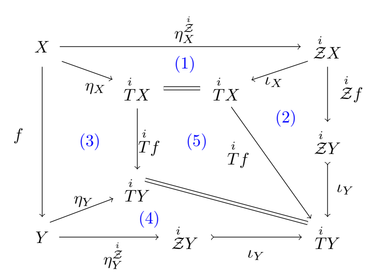
What's left is that $\tau_{W,X} \circ (W \otimes \iota_X)$ forms a central cone, it is proved by the following diagram:



(1), (3) α, ι_X are natural; (2) lemma 5; (4) ι_X is central; (5), (7) τ is natural; (6) T is a functor, $\alpha \circ \alpha^{-1} = id$, definition on graded strength; (8), (10), (11) definition of strength; (9) τ, τ' are natural; (12), (14) μ and α^{-1} are natural; (13) T is a functor, lemma 5; and rest are equalities.

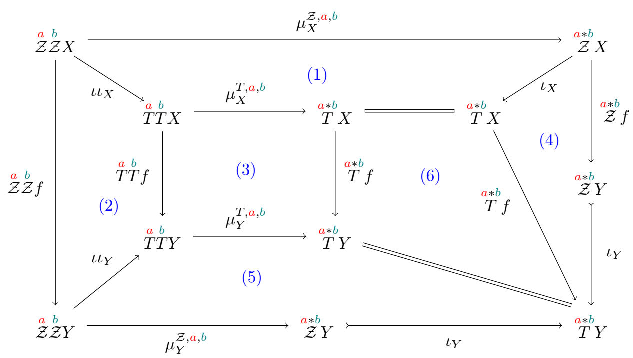
The rest of the proof is following the same strategy as that of centre Theorem in [2].

The last three definitions are exactly those of a morphism of strong graded monads (see Definition 7).
Using the fact that ι is monic (see Lemma 4), the following commutative diagram shows that $\eta^{\mathcal{Z}}$ is natural:



(1), (4) definition of $\eta^{\mathcal{Z}}$; (2) ι is natural; (3) η is natural; and (5) equality. Thus, we have proven that for any $f : X \to Y$, $\iota_Y \circ \mathcal{Z}f \circ \eta^{\mathcal{Z}}_X = \iota_Y \circ \eta^{\mathcal{Z}}_Y \circ f$. Besides, ι is monic, thus $\mathcal{Z}f \circ \eta^{\mathcal{Z}}_X = \eta^{\mathcal{Z}}_Y \circ f$ which proves that $\eta^{\mathcal{Z}}$ is natural. We will prove all the remaining diagrams with the same reasoning.

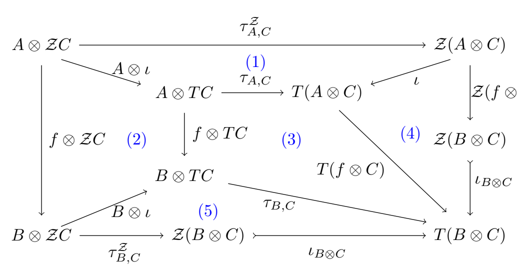
The following commutative diagram shows that $\mu^{\mathcal{Z}}$ is natural.



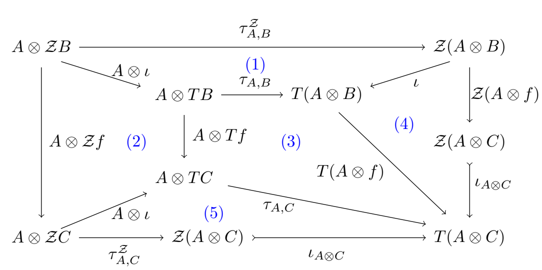
(1) (5) definition of $\mu^{\mathcal{Z}}$; (2) (4) ι is natural; (3) μ is natural and (6) equalities.

The commutative diagrams showing that $\tau^{\mathcal{Z}}$ is natural are just as those on normal monad by replacing $\mathcal{Z}$ as $\overset{z}{\mathcal{Z}}$, T as $\overset{z}{T}$ and ι as family of morphisms $\overset{z}{\iota} : \overset{z}{\mathcal{Z}} \to \overset{z}{T}$ indexed by elements in $\mathcal{G}$.

The following commutative diagrams show that $\tau^{\mathcal{Z}}$ is natural.

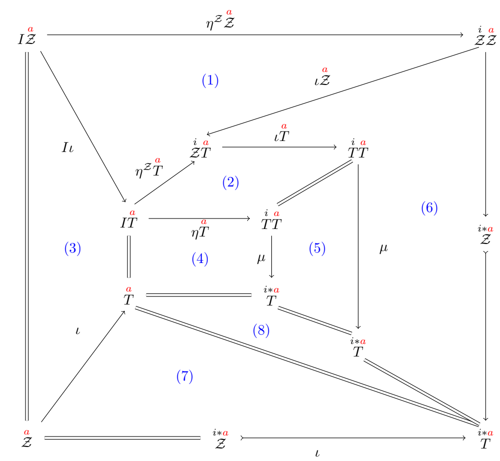


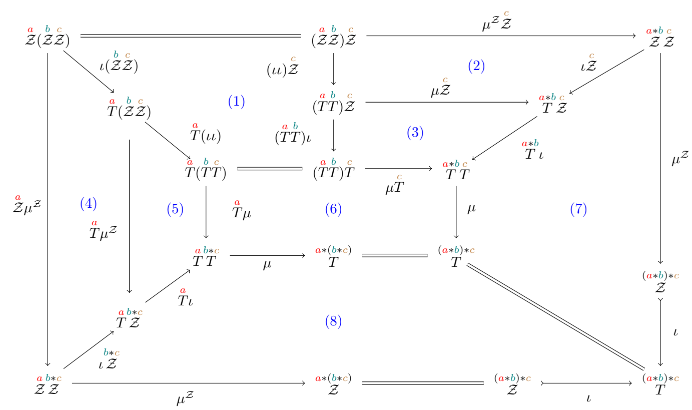
(1) definition of $\tau^{\mathcal{Z}}$, (2) ι is natural, (3) τ is natural, (4) ι is natural and (5) definition of $\tau^{\mathcal{Z}}$.



(1) definition of $\tau^{\mathcal{Z}}$, (2) ι is natural, (3) τ is natural, (4) ι is natural and (5) definition of $\tau^{\mathcal{Z}}$.

The following commutative diagrams prove that $\mathcal{Z}$ is a graded monad ($\overset{z}{\mathcal{Z}} \cdot I$ is omitted because it is very similar to $I \cdot \overset{z}{\mathcal{Z}}$).



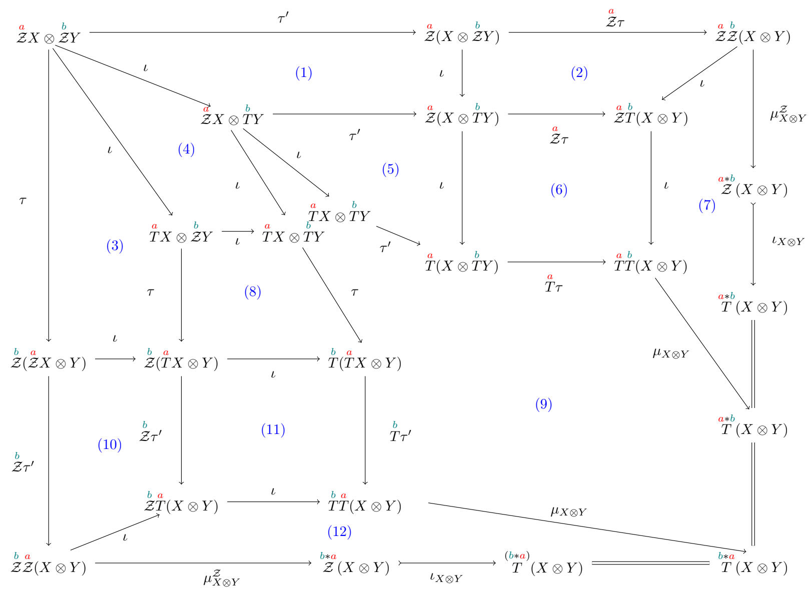


Top : (1) $\eta^{\mathcal{Z}}$ and ι are natural; (2) definition of $\eta^{\mathcal{Z}}$, (3) (5) (7) (8) equality; (4) definition of η and (6) definition of $\mu^{\mathcal{Z}}$.
Down : (1) equality; (2) definition of $\mu^{\mathcal{Z}}$, $\mathcal{Z}$ is a functor; (3) μ and ι are functors; (4) $\mu^{\mathcal{Z}}$ and ι are functors; (5) (7) (8) definition of $\mu^{\mathcal{Z}}$, T is a functor and (6) definition of graded monad.

Last Part:

$\mathcal{Z}$ is proven strong with very similar diagrams.
The following commutative diagram proves that $\mathcal{Z}$ is a commutative graded

monad:



(3)

(1) $\tau'^{\mathcal{Z}}$ is natural; (2) definition of $\tau^{\mathcal{Z}}$; (3) $\tau^{\mathcal{Z}}$ is natural; (4) **C** is monoidal; (5) definition of $\tau'^{\mathcal{Z}}$; (6) ι is natural; (7), (12) definition of $\mu^{\mathcal{Z}}$; (8) definition of $\tau^{\mathcal{Z}}$; (9) ι is central; (10) definition of $\tau'^{\mathcal{Z}}$; (11) ι is natural.