

Commutativity and Centre of Graded Monads

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Abstract

A monad can be seen as a special case of a graded monad. Effect-graded monads have been actively studied in recent years, they provide semantics for effects, in the style of Moggi’s monadic calculus, where effects are explicitly tracked [1]. “Grading” a monad with effects usually requires effect terms, the “gradations”, to have a monoidal structure. In practice, it is convenient to consider a pomonoid, which gives a poset over the gradations. Meanwhile, in the recent work by one of the supervisors [2], it is shown how the notion of “centre”, and more generally “centrality”, *i.e.*, the property for an effect to commute with all other effects, may be formulated for strong monads acting on symmetric monoidal categories. They characterize the existence of the centre of a strong monad (some of which relate it to the premonoidal centre [3] of Power and Robinson) and more generally, *central submonads*, which are necessarily commutative.

It was the question brought out by Flavien: what if we combine those two notions together and extend the centre notion from normal monads to the graded case? This extension aims to explore a more general sense of centre. The answer is yes, to achieve that, we need to first discuss the definitions on graded terms, most importantly the proper definition of morphisms between strong graded monads. Although the graded Kleisli construction is already defined in [4] (which is required if we want to construct a graded premonoidal centre), we have chosen not to go that path due to the extensive anticipated work which makes it unnecessary. Instead, we give supplementary proofs for one of the main theorems in [2] to circumvent reliance on the properties of premonoidal centre. Subsequently, we delve into the construction of *centre of graded monads* and *central graded submonads*.

SUMMARY (REQUIRED FOR MPRI REPORTS)

The general context

What is it about ? Where does it come from ? (See Abstract above)

What is the state of the art in this area ? (See Related Work II)

The research problem

What is the question that you studied ? Why is it important, what are the applications/consequences ? Is it a new problem ? If so, why are you the first researcher in the universe who consider it ? If not, why did you think that you could bring an original contribution ? (See Introduction I)

Your contribution

What is your solution to the question described in the last paragraph ?

Be careful, do *not* give technical details, only rough ideas !

Pay a special attention to the description of the *scientific* approach. (See Introduction I)

Arguments supporting its validity

What is the evidence that your solution is a good solution ? Experiments ? Proofs ?

Comment the robustness of your solution: how does it rely/depend on the working assumptions ? (See Introduction I)

Summary and future work

What is next ? In which respect is your approach general ? What did your contribution bring to the area ? What should be done now ? What is the good *next* question ? (See IX)

I. INTRODUCTION

The notion *centre* (or center) takes intuition from the *centre* of an algebraic object G . The general idea is that it is the collection of elements of G , which commute with all of its elements. It is denoted $Z(G)$, derived from the German term *Zentrum*. The centre has been defined for various structures such as groups, monoids, and certain categories. This intuition finds clear illustration in the case of centre $Z(R)$ of a ring R , which is defined as the multiplicative subset consisting of all elements $s \in R$, s.t. $r \cdot s = s \cdot r$ holds true. This gives a property within a computational context, it means that the sequence of evaluating two elements does not impact the overall computation. To engage with computational effects, the concept of monads becomes essential.

A monad is a structure similar to monoid, but lives in bicategory instead of a monoidal category. Actually, quoting from Saunders Mac Lane [5]: *A monad in category X is just a monoid in the category of endofunctors of X , with product x replaced by composition of endofunctors and unit set by the identity endofunctor.* Moggi demonstrated in [6], [7] that the monads give a model of computational effects, which now has an important role in programming languages, especially functional ones such as Haskell.

Now if we evolve the ring to a graded ring, we have the concept of graded monads. Let $(R, +, 0, \times, 1)$ be a ring and N be the additive monoid on natural numbers. An N -grading on R is an N -indexed family $(A_m)_{m \in N}$ of Abelian groups s.t. $R = \bigoplus_{m \in N} A_m$ and the following holds for all $m, n \in N$: $\{x \times y \mid x \in A_m, y \in A_n\} \subseteq A_{m+n}$ [4]. Graded monads keep track of side effects with help of gradations, normally a monoidal category. In fact, it's a more general notion – if we take the gradations as terminal monoidal category 1 , we obtain a normal monad.

To answer the question in summary: Is it a new problem? If so, why are you the first researcher in the universe who consider it? [2] is the first time we know to talk about centre notion of endofunctor (monad), and construct computational meaning base on this notion. This work is the first time we know discussing the centre notion of lax monoidal functor (graded monad), which is a functor between different structures. Centre relies on commutativity, there is rarely any discussions so far even on commutativity of graded monads, so I do think I'm the first in the universe who consider it.

My contributions are mainly in three parts:

- First, supplementary proof of one of the main Theorems of [2] 17 without using *premonoidal centre*. More specifically, [2] introduces the centre of monads on a monoidal symmetric category, with help of the fact that *central cone* it defined being an equivalent notion of *central morphism* via the *premonoidal centre* of [3]. The reason we need this proof is that, the construction of premonoidal centre relies on Kleisli category, but that of graded monads is too complicated and loses the point of using it. Since it is no longer trivial for the morphisms to be central, I have to prove them in V. Although it is already a complicated proof, it still is the easiest way for the monadic structure constructions to be further extended to graded case. With this proof, the centre notion on monads of [2] no longer depends on the central morphism defined in [3].
- Then the proper graded definitions in IV: the graded monads on different gradations, the graded strength, properties on graded co-strength. Most importantly, the Definition 10: commutativity of graded monad and Definition 11: morphisms between strong graded monads. They are the foundations for the construction later, we had many attempts on these definitions, majority of them did not work out. Although it has a relatively small partition within this report, is it one of the most time-consuming parts in this internship.
- With the definitions above, I am able to present and prove our main Theorem 24. The construction on the monadic part is necessarily similar to [2], by the fact that they are both functors (one being endofunctors and another being lax monoidal functors). However, it is not at all as easy as just adding the gradations and gradings, since now it is a functor between different structures. New lemmas, theorems, propositions on the new definitions need to be checked and proved, e.g. definition of a graded monad is very different from a normal monad. Furthermore, this theorem leads to a full notion of the central graded submonads. As the examples I give in VII-A, this notion shows great potential in computations.

II. RELATED WORK

This internship takes intuitions from works of the supervisors [2] and [8]. In fact, for the construction on monadic part of the centre of graded monads, it is necessary to use similar constructions as in [2], including the similar theorems, definition of graded central cone, and proof strategy of the main Theorem V. Another related work concerning the monadic part is [3], they give the *central morphism* and *premonoidal centre*. Nonetheless, our approach in this part has no connection with that as we deliberately avoid the utilization of Kleisli constructions.

To construct the centre, we have to first study commutativity. [9] describes a general framework for commutativity based on enriched category theory, and preprint [10] gives definition of commutative graded monads, and it is as far as I know the only work yet discussing commutativity of graded monads. However, its definition relies on their previous work of *suspension* of a monoidal category.

For the gradation part, [8] defined graded monads as lax monoidal functors between skew monoidal categories, which is more general than those between monoidal categories due to less monoidal property requires, and defined the morphisms between them. It exceeds the scope of our work since our focus so far being monoid graded monads on monoidal category, but it is where we draw inspiration for the definition of the morphisms between strong graded monads. While other definitions exist for morphisms between graded monads, many of them lack the required generality to be applicable to our context. For example [11] gives morphisms and transfers between graded monads that are graded by same category. But they do not give the definition of strong graded monads, nor the morphisms between different gradations. Our construction of morphism between strong graded monads in Definition 11 is novel and the only work I know so far using homomorphism between gradations.

III. BACKGROUND

A. Monoids and Monads

Here we recall the definition of a monoid, with which we can introduce the notion of a monad.

Definition 1 (Monoids and Homomorphisms between Monoids). A *monoid* $(M, i, *)$ is a set M together with a binary operation $M \times M \rightarrow M$ denote as $*$, and neutral element i , s.t. for all $x, y, z \in M$:

- Associative law : $(x * y) * z = x * (y * z)$;
- Left and right unit laws: $i * x = x = x * i$.

Given two monoids $(M, i, *)$ and (M', i', \otimes) , a monoid homomorphism $\phi : M \rightarrow M'$ between them is a map that preserves $*$ operation and i s.t.:

- $\phi i = i'$,
- $\phi(x * y) = \phi x' \otimes \phi y'$.

Assuming the reader's familiarity with the concept of a category, a monad can be understood as a monoid in the category of endofunctors. The term "monad" originates from the fusion of "monoid" and "triad", signifying its nature as both a triple (consisting of one functor and two transformations) and a monoidic structure. The definition of a monad can be formulated by substituting the associative and unit laws:

Definition 2 (Monad). Given a category \mathbf{C} , a *monad* is an endofunctor $\mathcal{T} : \mathbf{C} \rightarrow \mathbf{C}$ equipped with two natural transformations:

- The *unit* $\eta : Id \Rightarrow \mathcal{T}$
- The *multiplication* $\mu : \mathcal{T}^2 \Rightarrow \mathcal{T}$

s.t. the following diagrams commute:

$$\begin{array}{ccc} \mathcal{T}^3 X & \xrightarrow{\mu_{\mathcal{T}X}} & \mathcal{T}^2 X \\ \mathcal{T}\mu_X \downarrow & & \downarrow \mu_X \\ \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{T}X \end{array} \quad \begin{array}{ccc} \mathcal{T}X & \xrightarrow{\mathcal{T}\eta_X} & \mathcal{T}^2 X \\ \eta_{\mathcal{T}X} \downarrow & \searrow & \downarrow \mu_X \\ \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{T}X \end{array}$$

B. Monoidal Category, Strong and Commutative Monads

Recall the definition of a *monoidal category*:

Definition 3 (Monoidal Category). A monoidal category $\mathbf{C} : (\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ is a category equipped with a monoidal structure, defined as:

- The *monoidal product* or *tensor product*, which is a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$;
- The *monoidal unit*, *unit object*, or *identity object* I ;
- Three natural isomorphisms:
 - The *associator* $\alpha_{W,X,Y} : W \otimes (X \otimes Y) \cong (W \otimes X) \otimes Y$;
 - The *left unitor* $\lambda_X : I \otimes X \cong X$ and *right unitor* $\rho_X : X \otimes I \cong X$;

s.t. the following diagrams expressing the coherence conditions commute:

$$\begin{array}{ccc} X \otimes (I \otimes Y) & \xrightarrow{\alpha_{X,I,Y}} & (X \otimes I) \otimes Y \\ \downarrow X \otimes \lambda_Y & & \downarrow \rho_X \otimes Y \\ X \otimes Y & & X \otimes Y \end{array} \quad \begin{array}{ccc} V \otimes (W \otimes (X \otimes Y)) & \xrightarrow{\alpha_{V,W,X \otimes Y}} & (V \otimes W) \otimes (X \otimes Y) \xrightarrow{\alpha_{V \otimes W, X, Y}} & ((V \otimes W) \otimes X) \otimes Y \\ \downarrow V \otimes \alpha_{W,X,Y} & & \downarrow \alpha_{V,W,X} \otimes Y \\ V \otimes (W \otimes X) \otimes Y & \xrightarrow{\alpha_{V,W \otimes X, Y}} & (V \otimes (W \otimes X)) \otimes Y \end{array}$$

The next diagrams in this subsection are directly from [2].

Next, we introduce an additional structure, the *monadic strength*, which requires that the basic category to be a monoidal category. It ensures that the monad interacts appropriately with the monoidal structure of the base category. It was discovered in [2] to be the main object of study in terms of the commutativity and construction of the centre.

Definition 4 (Strong Monad). Given a monoidal category $\mathbf{C} : (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$, a *strong monad* is a monad (\mathcal{T}, η, μ) equipped with a natural transformation, *strength*, $\tau_{X,Y} : X \otimes \mathcal{T}Y \rightarrow \mathcal{T}(X \otimes Y)$, s.t. for all objects X, Y in \mathbf{C} , the following diagrams commute:

$$\begin{array}{ccc}
 I \otimes \mathcal{T}X & \xrightarrow{\tau_{I,X}} & \mathcal{T}(I \otimes X) \\
 \searrow \lambda_{\mathcal{T}X} & & \downarrow \mathcal{T}\lambda_X \\
 & & \mathcal{T}X
 \end{array}
 \quad
 \begin{array}{ccc}
 X \otimes Y & \xrightarrow{X \otimes \eta_Y} & X \otimes \mathcal{T}Y \\
 \searrow \eta_{X \otimes Y} & & \downarrow \tau_{X,Y} \\
 & & \mathcal{T}(X \otimes Y)
 \end{array}$$

$$\begin{array}{ccc}
 (W \otimes X) \otimes \mathcal{T}Y & \xrightarrow{\tau_{W \otimes X, Y}} & \mathcal{T}((W \otimes X) \otimes Y) \\
 \downarrow \alpha_{W, X, \mathcal{T}Y} & & \downarrow \mathcal{T}\alpha_{W, X, Y} \\
 W \otimes (X \otimes \mathcal{T}Y) & \xrightarrow[W \otimes \tau_{X, Y}]{} & W \otimes \mathcal{T}(X \otimes Y) \xrightarrow[\tau_{W, X \otimes Y}]{} \mathcal{T}(W \otimes (X \otimes Y))
 \end{array}$$

$$\begin{array}{ccc}
 X \otimes \mathcal{T}^2 Y & \xrightarrow{\tau_{X, \mathcal{T}Y}} & \mathcal{T}(X \otimes \mathcal{T}Y) \xrightarrow{\mathcal{T}\tau_{X, Y}} & \mathcal{T}^2(X \otimes Y) \\
 \downarrow X \otimes \mu_Y & & & \downarrow \mu_{X \otimes Y} \\
 X \otimes \mathcal{T}Y & \xrightarrow{\tau_{X, Y}} & \mathcal{T}(X \otimes Y)
 \end{array}$$

We can now introduce the definition of a *commutative monad*, which enjoys stronger coherence properties than a strong monad.

Before that we also need a notion of *costrength*. Given a *symmetric monoidal category* $(\mathbf{C}, \otimes, I, \gamma)$, for all objects X, Y in \mathbf{C} , the *costrength* $\tau'_{X,Y} : \mathcal{T}X \otimes Y \rightarrow \mathcal{T}(X \otimes Y)$ is defined as $\tau'_{X,Y} \stackrel{\text{def}}{=} \mathcal{T}(\gamma_{Y,X}) \circ \tau_{Y,X} \circ \gamma_{\mathcal{T}X, Y}$.

Definition 5 (Commutative Monad). Let $(\mathcal{T}, \eta, \mu, \tau)$ be a strong monad on a *symmetric monoidal category* $(\mathbf{C}, \otimes, I, \gamma)$. Then, \mathcal{T} is said to be *commutative* if the following diagram commutes for all objects X, Y in \mathbf{C} :

$$\begin{array}{ccc}
 \mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau_{\mathcal{T}X, Y}} & \mathcal{T}(\mathcal{T}X \otimes Y) \xrightarrow{\mathcal{T}\tau'_{X, Y}} & \mathcal{T}^2(X \otimes Y) \\
 \downarrow \tau'_{X, \mathcal{T}Y} & & & \downarrow \mu_{X \otimes Y} \\
 \mathcal{T}(X \otimes \mathcal{T}Y) & \xrightarrow[\mathcal{T}\tau_{X, Y}]{} & \mathcal{T}^2(X \otimes Y) \xrightarrow[\mu_{X \otimes Y}]{} & \mathcal{T}(X \otimes Y)
 \end{array}$$

Then we define a morphism of strong monads:

Definition 6 (Morphism of Strong Monads [12]). Given two strong monads $(\mathcal{T}, \eta^{\mathcal{T}}, \mu^{\mathcal{T}}, \tau^{\mathcal{T}})$ and $(\mathcal{P}, \eta^{\mathcal{P}}, \mu^{\mathcal{P}}, \tau^{\mathcal{P}})$ over a category \mathbf{C} , then a *morphism of strong monads* is a natural transformation $\iota : \mathcal{T} \Rightarrow \mathcal{P}$ that makes the following diagrams commute:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 X & & X \\
 \eta_X^{\mathcal{T}} \swarrow & & \searrow \eta_X^{\mathcal{P}} \\
 \mathcal{T}X & \xrightarrow{\iota_X} & \mathcal{P}X
 \end{array}
 &
 \begin{array}{ccc}
 X \otimes \mathcal{T}Y & \xrightarrow{X \otimes \iota_Y} & X \otimes \mathcal{P}Y \\
 \tau_{X, Y}^{\mathcal{T}} \downarrow & & \downarrow \tau_{X, Y}^{\mathcal{P}} \\
 \mathcal{T}(X \otimes Y) & \xrightarrow[\iota_{X \otimes Y}]{} & \mathcal{P}(X \otimes Y)
 \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{T}^2 X & \xrightarrow{\iota_{\mathcal{T}X}} & \mathcal{P}\mathcal{T}X \xrightarrow{\mathcal{P}\iota_X} & \mathcal{P}^2 X \\
 \mu_X^{\mathcal{T}} \downarrow & & & \downarrow \mu_X^{\mathcal{P}} \\
 \mathcal{T}X & \xrightarrow{\iota_X} & \mathcal{P}X
 \end{array}$$

Strong monads over \mathbf{C} and the strong monad morphisms form a category $\mathbf{StrMnd}(\mathbf{C})$. We call \mathcal{T} a *strong submonad* of \mathcal{P} and ι a *submonad morphism*, if ι is a monomorphism in $\mathbf{StrMnd}(\mathbf{C})$.

We omit the Kleisli structure and premonoidal structure of strong monads. Although they are essential for the construction of centre notion and theorem 17 in [2], we do not use them on graded monads.

IV. NEW DEFINITIONS ON GRADED MONADS

In the beginning of the internship, we approached the problem by considering a special instance of graded monad which is a strict monoidal functor to the bicategory. However, we later proved the results on the lax monoidal functor which is a full version of graded monad. Hence, only the full version of graded monads is introduced in this report.

The definitions of grade monads vary noticeably from each other and could potentially lead to confusion, if not being careful with the selection of gradations. Given this circumstance, it's necessary to discuss them before going further.

A. Discussion on Definitions of Graded Monads

To be able to operate multiplication on the gradations, most of the discussions on graded monads consider monoidal category as gradations, instead of any category. In fact, we can even consider a skew monoidal category [8].

Throughout this work, we choose to use the monoid graded monads for the centre construction of graded monads. This choice is driven by the fact that monoid is the one of the simplest monoidal structures which has a known centre. More complex gradations, for example as a monoidal category, pomonoid, or even skew monoidal category, could be left for future work base on this construction.

Definition 7 (Monoid Graded Monads). Let $\mathcal{G} : (G, i, *)$ be a monoid. A \mathcal{G} -graded monad on a category \mathbf{C} (not necessary monoidal) is:

- for any $a \in G$, a functor $\mathcal{T}^a : \mathbf{C} \rightarrow \mathbf{C}$;
- a natural transformation $\eta : Id \rightarrow \mathcal{T}^i$;
- for any $a, b \in G$, a natural transformation $\mu^{a,b} : \mathcal{T}^a \cdot \mathcal{T}^b \rightarrow \mathcal{T}^{a*b}$, (the \cdot here is functor composition, and we can omit the gradations on μ for convenience);

s.t. the following diagrams commute:

$$\begin{array}{ccc}
 Id \cdot \mathcal{T}^a & \xrightarrow{\eta \cdot \mathcal{T}^a} & \mathcal{T}^i \cdot \mathcal{T}^a \\
 \parallel & & \downarrow \mu^{i,a} \\
 \mathcal{T}^a & \xlongequal{\quad} & \mathcal{T}^{i*a}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{T}^a \cdot Id & \xrightarrow{\mathcal{T}^a \cdot \eta} & \mathcal{T}^a \cdot \mathcal{T}^i \\
 \parallel & & \downarrow \mu^{a,i} \\
 \mathcal{T}^a & \xlongequal{\quad} & \mathcal{T}^{a*i}
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{T}^a \cdot (\mathcal{T}^b \cdot \mathcal{T}^c) & \xlongequal{\quad} & (\mathcal{T}^a \cdot \mathcal{T}^b) \cdot \mathcal{T}^c \xrightarrow{\mu^{a,b} \cdot \mathcal{T}^c} \mathcal{T}^{a*b} \cdot \mathcal{T}^c \\
 \downarrow \mathcal{T}^a \cdot \mu^{b,c} & & \downarrow \mu^{a*b,c} \\
 \mathcal{T}^a \cdot \mathcal{T}^{b*c} & \xrightarrow{\mu^{a,b*c}} & \mathcal{T}^{a*(b*c)} \xlongequal{\quad} \mathcal{T}^{(a*b)*c}
 \end{array}$$

In short, a graded monad on \mathbf{C} is a lax monoidal functor as a discrete monoidal category from the category of gradations (this time the monoid \mathcal{G}) to the endofunctor category of \mathbf{C} , $([\mathbf{C}, \mathbf{C}], Id_{\mathbf{C}}, \cdot)$:

$$(\mathcal{G}, i, *) \rightarrow ([\mathbf{C}, \mathbf{C}], Id_{\mathbf{C}}, \cdot).$$

Note that the $Id_{\mathbf{C}}$ here is the identity in category \mathbf{C} .

There also exists different definitions of graded monads depending on the perspective. For instance, in Appendix Definition 30, we can choose an ordered grading which correspond to a thin category on pomonoid as gradations. Grading with a pomonoid is more general than monoid, one can see monoid as a special instance of pomonoid wherein the order is trivial, and its ordered structure is very useful in computational aspect.

B. New Definitions on Graded Monads

Definition 8 (Strong Graded Monad). Let $\mathcal{G} : (G, i, *)$ be a monoid. A *strong \mathcal{G} -graded monad* over a monoidal category $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ is a \mathcal{G} -graded monad (\mathcal{T}, η, μ) equipped with a family of natural transformations $\tau_{X,Y}^a : X \otimes^a \mathcal{T}Y \rightarrow \mathcal{T}(X \otimes Y)$ indexed by elements in \mathcal{G} (index of τ can be omitted because it always stays same with \mathcal{T}), called *graded strength*, s.t. for any object b in G and objects X, Y in \mathbf{C} , the following diagrams commute:

$$\begin{array}{ccc}
 I \otimes^a \mathcal{T}X & \xrightarrow{\tau_{I,X}^a} & \mathcal{T}(I \otimes X) \\
 \searrow \lambda_{\mathcal{T}X}^a & & \downarrow \mathcal{T}\lambda_X^a \\
 & & \mathcal{T}X
 \end{array}
 \qquad
 \begin{array}{ccc}
 X \otimes Y & \xrightarrow{X \otimes \eta_Y} & X \otimes^i \mathcal{T}Y \\
 \searrow \eta_{X \otimes Y} & & \downarrow \tau_{X,Y} \\
 & & \mathcal{T}(X \otimes Y)
 \end{array}$$

$$\begin{array}{ccc}
 (W \otimes X) \otimes^a \mathcal{T}Y & \xrightarrow{\tau_{W \otimes X, Y}^a} & \mathcal{T}((W \otimes X) \otimes Y) \\
 \downarrow \alpha_{W, X, \mathcal{T}Y}^a & & \downarrow \mathcal{T}\alpha_{W, X, Y}^a \\
 W \otimes (X \otimes^a \mathcal{T}Y) & \xrightarrow{W \otimes \tau_{X, Y}^a} & W \otimes \mathcal{T}(X \otimes Y) \xrightarrow{\tau_{W, X \otimes Y}^a} \mathcal{T}(W \otimes (X \otimes Y))
 \end{array}$$

$$\begin{array}{ccc}
 X \otimes (\mathcal{T} \cdot \mathcal{T}Y) & \xrightarrow{\tau_{X, \mathcal{T}Y}^a} & \mathcal{T}(X \otimes \mathcal{T}Y) \xrightarrow{\mathcal{T}\tau_{X, Y}^a} \mathcal{T} \cdot \mathcal{T}(X \otimes Y) \\
 \downarrow X \otimes \mu_Y^{a,b} & & \downarrow \mu_{X \otimes Y}^{a,b} \\
 X \otimes^a \mathcal{T}Y & \xrightarrow{\tau_{X, Y}^a} & \mathcal{T}(X \otimes Y)
 \end{array}$$

Similar to normal monad, we can define the *\mathcal{G} -graded costrength* $\tau'_{X,Y}^a : \mathcal{T}X \otimes Y \rightarrow \mathcal{T}(X \otimes Y)$ of \mathcal{G} -graded \mathcal{T} on a *symmetric* monoidal category $(\mathbf{C}, \otimes, I, \gamma)$ as $\tau'_{X,Y}^a \stackrel{\text{def}}{=} \mathcal{T}(\gamma_{Y,X}) \circ \tau_{Y,X}^a \circ \gamma_{\mathcal{T}X, Y}^a$. We also omit the gradations on it for convenience.

Proposition 9 (Coherence Properties of costrength). *For all elements a, b in G , X, Y in \mathbf{C} , the following diagrams commute:*

$$\begin{array}{ccc}
 \mathcal{T}X \otimes I & \xrightarrow{\tau'_{X,I}} & \mathcal{T}(X \otimes I) \\
 \searrow \rho_{\mathcal{T}X}^a & & \downarrow \mathcal{T}\rho_X^a \\
 & & \mathcal{T}X
 \end{array}
 \qquad
 \begin{array}{ccc}
 X \otimes Y & \xrightarrow{\eta_X \otimes Y} & \mathcal{T}X \otimes Y \\
 \searrow \eta_{X \otimes Y} & & \downarrow \tau'_{X, Y} \\
 & & \mathcal{T}(X \otimes Y)
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{T}W \otimes (X \otimes Y) & \xrightarrow{\tau'_{W, X \otimes Y}} & \mathcal{T}(W \otimes (X \otimes Y)) \\
 \downarrow \alpha_{\mathcal{T}W, X, Y}^a & & \downarrow \mathcal{T}\alpha_{W, X, Y}^a \\
 (\mathcal{T}W \otimes X) \otimes Y & \xrightarrow{\tau'_{W, X} \otimes Y} & \mathcal{T}(W \otimes X) \otimes Y \xrightarrow{\tau'_{W \otimes X, Y}} \mathcal{T}((W \otimes X) \otimes Y)
 \end{array}$$

$$\begin{array}{ccc}
 (\mathcal{T} \cdot \mathcal{T}X) \otimes Y & \xrightarrow{\tau'_{\mathcal{T}X, Y}} & \mathcal{T}(\mathcal{T}X \otimes Y) \xrightarrow{\mathcal{T}\tau'_{X, Y}} \mathcal{T} \cdot \mathcal{T}(X \otimes Y) \\
 \downarrow \mu_X^{a,b} \otimes Y & & \downarrow \mu_{X \otimes Y}^{a,b} \\
 \mathcal{T}X \otimes Y & \xrightarrow{\tau'_{X, Y}} & \mathcal{T}(X \otimes Y)
 \end{array}$$

Proof. Only η and μ part of this proposition is going to be used in the proofs later, hence we only give proofs on those two parts, the rest could be proved in a similar way.

The proof of η :

$$\begin{array}{ccc}
 X \otimes Y & \xrightarrow{\eta_{X \otimes Y}} & \mathcal{T}X \otimes Y \\
 \downarrow \gamma_{X,Y} & \searrow \gamma_{\mathcal{T}X,Y} & \downarrow \tau'_{X,Y} \\
 Y \otimes X & \xrightarrow{Y \otimes \eta_X} & Y \otimes \mathcal{T}X \\
 \downarrow \gamma_{Y,X} & \searrow \eta_{Y \otimes X} & \downarrow \tau_{Y,X} \\
 X \otimes Y & \xrightarrow{\eta_{X \otimes Y}} & \mathcal{T}(X \otimes Y)
 \end{array}$$

(1) γ is natural; (2) $\gamma_{X,Y}; \gamma_{Y,X} = id$; (3) definition of τ ; (4) definition of τ' ; and (5) η and γ are natural.

(1) γ is natural; (2) $\gamma_{X,Y}; \gamma_{Y,X} = id$; (3) definition of τ ; (4) definition of τ' ; and (5) η and γ are natural.
 μ :

$$\begin{array}{ccccc}
 (\mathcal{T} \cdot \mathcal{T}X) \otimes Y & \xrightarrow{\tau'_{\mathcal{T}X,Y}} & \mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X,Y}} & \mathcal{T} \cdot \mathcal{T}(X \otimes Y) \\
 \downarrow \gamma_{\mathcal{T} \cdot \mathcal{T}X,Y} & \searrow \gamma_{\mathcal{T}X,Y} & \downarrow \mathcal{T}\tau_{Y,\mathcal{T}X} & \searrow \mathcal{T} \cdot \mathcal{T}\tau_{Y,X} & \downarrow \mu_{X \otimes Y}^{a,b} \\
 Y \otimes (\mathcal{T} \cdot \mathcal{T}X) & \xrightarrow{\tau_{Y,\mathcal{T}X}} & \mathcal{T}(Y \otimes \mathcal{T}X) & \xrightarrow{\mathcal{T}\tau_{Y,X}} & \mathcal{T} \cdot \mathcal{T}(Y \otimes X) \\
 \downarrow \gamma_{Y,\mathcal{T}X} & \searrow \mu_{Y \otimes X}^{a,b} & \downarrow \tau_{Y,X} & \searrow \mu_{Y \otimes X}^{a,b} & \downarrow \mathcal{T} \cdot \mathcal{T}\gamma_{Y,X} \\
 X \otimes \mu_Y^{a,b} & \xrightarrow{\gamma_{X,Y}^{a,b}} & Y \otimes \mathcal{T}X & \xrightarrow{\tau_{Y,X}} & \mathcal{T}(Y \otimes X) \\
 \downarrow \gamma_{X,Y}^{a,b} & \searrow \mu_{X \otimes Y}^{a,b} & \downarrow \tau_{Y,X} & \searrow \mu_{X \otimes Y}^{a,b} & \downarrow \mathcal{T}\gamma_{Y,X} \\
 \mathcal{T}X \otimes Y & \xrightarrow{\tau'_{X,Y}} & \mathcal{T}(X \otimes Y) & & \mathcal{T}(X \otimes Y)
 \end{array}$$

(1) fact that $\mathcal{T}\gamma_{\mathcal{T}X,Y}^a = \mathcal{T}\gamma_{Y,\mathcal{T}X}^{-1}$, and definition of τ' ; (2) \mathcal{T} is a functor and definition of τ' ; (3) γ is natural; (4) definition of strength; (5) $\mu_{X \otimes Y}^{a,b}$ and γ are natural and (6) definition of τ' . □

With graded strength and costrength, we can then define commutative in the graded monad case.

Definition 10 (Commutative Graded Monad). Let $\mathcal{G} : (G, i, *)$ be a commutative monoid. Let $(\mathcal{T}, \eta, \mu, \tau)$ be a strong \mathcal{G} -graded monad on a symmetric monoidal category $(\mathbf{C}, \otimes, I, \gamma)$. Then, \mathcal{T} is said to be *commutative* if for any $a, b \in G$, and any object $X, Y \in \mathbf{C}$, the following diagram commutes:

$$\begin{array}{ccccc}
 \mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau_{\mathcal{T}X,Y}^a} & \mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X,Y}^b} & \mathcal{T} \cdot \mathcal{T}(X \otimes Y) \\
 \downarrow \tau'_{X,\mathcal{T}Y} & & & & \downarrow \mu_{X \otimes Y}^{b,a} \\
 \mathcal{T}(X \otimes \mathcal{T}Y) & \xrightarrow{\tau_{X,\mathcal{T}Y}^a} & \mathcal{T} \cdot \mathcal{T}(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}^{a,b}} & \mathcal{T}(X \otimes Y)
 \end{array}$$

C. Discussion on Morphisms between Graded Monads

One of the major challenges of this internship is to figure out a proper definition of the morphisms between strong graded monads. This task is of paramount importance as it underpins subsequent constructions for the later constructions on the centre. We spent weeks discussing and refining this part. It is a complex and delicate problem and requires a lot of attempts and works in the detail. While we explored certain scenarios that appeared more intricate, it eventually turns out to be simpler.

Intuitively this morphism between strong graded monads is needed as the relation between the centre of the strong graded monad and the original strong graded monad, also essential for the construction of the graded submonad. To achieve the construction of such a morphism, we need morphisms from both gradations and the strong monads. The morphisms between strong monads are already provided in Definition 6, the remaining task involves selecting and defining the appropriate morphisms between gradations.

In our initial attempt, we began the internship by considering the gradations enjoy equalities. However, it has come to our understanding that the morphism between the graded monads should not be treated as a strict functor, as initially presumed. Rather, it should be a lax functor, as explained in [8].

Subsequently, I tried to use the morphisms between different monoids from the same monoidal category, as outlined in [https://en.wikipedia.org/wiki/Monoid_\(category_theory\)](https://en.wikipedia.org/wiki/Monoid_(category_theory)). However, the gradations of this approach are different kinds of monoids (the monoids we use in this work for gradations are general monoids, not monoids in monoidal category), which raised new problems:

- Whether a centre could be constructed on the monoids of monoidal category remains uncertain, hence unsuitable for the later construction of the centre on gradations.
- It leaves more relations undefined, for example the proper definition of the arrow $\mathcal{P} \rightarrow \mathcal{P}$ is unknown.
- It still does not solve the problem when two gradations could be coming from different categories, e.g. have different i and $*$.

Finally, with help of Vladimir, I realize that we could define a homomorphism between monoids, as given in Definition 1. This approach solves the problem immediately.

Definition 11 (Morphism between Strong Graded Monads). Let \mathcal{G} and \mathcal{G}' be two monoids $\mathcal{G} : (G, i, *)$, $\mathcal{G}' : (G', i', \otimes)$. Let $(\mathcal{T}, \eta^{\mathcal{T}}, \mu^{\mathcal{T}}, \tau^{\mathcal{T}})$ be a \mathcal{G} -graded strong monad and $(\mathcal{P}, \eta^{\mathcal{P}}, \mu^{\mathcal{P}}, \tau^{\mathcal{P}})$ a \mathcal{G}' -graded strong monad over a symmetric monoidal category $\mathbf{C} : (C, \otimes, I, \gamma)$.

Then a *morphism of strong graded monads* is a family of natural transformations between two graded monads together with

- A monoid homomorphism as in Definition 1 from \mathcal{G} to $\mathcal{G}' : \phi : \mathcal{G} \rightarrow \mathcal{G}'$,
- A family of natural transformations between two endofunctors $\overset{a}{\iota} : \overset{a}{\mathcal{T}} \Rightarrow \overset{\phi a}{\mathcal{P}}$ indexed by the elements of \mathcal{G} ,¹

s.t. for all a, b in \mathcal{G} and X, Y in \mathbf{C} , the following diagrams commute:

$$\begin{array}{ccc}
 & X & \\
 \eta_X^{\mathcal{T}} \swarrow & & \searrow \eta_X^{\mathcal{P}} \\
 \overset{i}{\mathcal{T}}X & & \overset{i'}{\mathcal{P}}X \\
 \downarrow \iota_X & & \parallel \\
 & \overset{\phi i}{\mathcal{P}}X &
 \end{array}
 \qquad
 \begin{array}{ccc}
 X \otimes \overset{a}{\mathcal{T}}Y & \xrightarrow{X \otimes \iota_Y} & X \otimes \overset{\phi a}{\mathcal{P}}Y \\
 \tau_{X,Y}^{\mathcal{T}} \downarrow & & \downarrow \tau_{X,Y}^{\mathcal{P}} \\
 \overset{a}{\mathcal{T}}(X \otimes Y) & \xrightarrow{\iota_{X \otimes Y}} & \overset{\phi a}{\mathcal{P}}(X \otimes Y) \\
 \\
 \overset{a}{\mathcal{T}} \cdot \overset{b}{\mathcal{T}}X & \xrightarrow{\overset{\iota_b}{\mathcal{T}}X} \overset{\phi a}{\mathcal{P}} \cdot \overset{b}{\mathcal{T}}X \xrightarrow{\overset{\phi a}{\mathcal{P}}\iota_X} \overset{\phi a}{\mathcal{P}} \cdot \overset{\phi b}{\mathcal{P}}X & \\
 \mu_X^{\mathcal{T}} \downarrow & & \downarrow \mu_X^{\mathcal{P}} \\
 \overset{a*b}{\mathcal{T}}X & \xrightarrow{\iota_X} \overset{\phi(a*b)}{\mathcal{P}}X & \parallel \overset{\phi a \otimes \phi b}{\mathcal{P}}X
 \end{array}$$

The subtlety for this morphism is to explain the relations between $\overset{i'}{\mathcal{P}}, \overset{\phi i}{\mathcal{P}}$ and $\overset{\phi a \otimes \phi b}{\mathcal{P}}, \overset{\phi(a*b)}{\mathcal{P}}$, which turn out to be equalities due to the monoid homomorphism: $\phi i = i'$ and $\phi(a * b) = \phi a \otimes \phi b$. From those equalities we can get $\overset{\phi i}{\mathcal{P}}X = \overset{i'}{\mathcal{P}}X$ and $\overset{\phi(a*b)}{\mathcal{P}}X = \overset{\phi a \otimes \phi b}{\mathcal{P}}X$.

¹The index of ι is omitted for convenience because it always stays same with the left element of the arrow.

V. THE CENTRE OF A STRONG MONAD

In the first half of this section, we will provide a concise overview of the construction of centre of a strong monad, as detailed in [2], by the novel notion of *central cone*, as well as its relation with *premonoidal centre*. In the second half, I give my supplementary proof for theorem 17, without using proposition 14.

First, we construct the centre for any (necessarily strong) monad on **Set**, which is a special case of the general case.

Definition 12 (Centre on **Set** [2]). Given a strong monad $(\mathcal{T}, \eta, \mu, \tau)$ on **Set** with right strength τ' , we say that the *centre* of \mathcal{T} at X , written $\mathcal{Z}X$, is the set

$$\mathcal{Z}X \stackrel{\text{def}}{=} \{t \in \mathcal{T}X \mid \forall Y \in \text{Ob}(\mathbf{Set}), \forall s \in \mathcal{T}Y, \mu(\mathcal{T}\tau'(\tau(t, s))) = \mu(\mathcal{T}\tau(\tau'(t, s)))\}.$$

We write $\iota_X : \mathcal{Z}X \subseteq \mathcal{T}X$ for the indicated subset inclusion.

In other words, the centre of \mathcal{T} at X is the subset of $\mathcal{T}X$ which contains all monadic elements for which (5) holds when the set X is fixed and the set Y ranges over all sets.

Throughout the remainder of the section, we assume we are given a symmetric monoidal category $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ with an inverse operation of α as α^{-1} , and a strong monad $(\mathcal{T}, \eta, \mu, \tau)$ on it with right strength τ' .

In **Set**, the centre is defined through the subsets of $\mathcal{T}X$ which only contain elements that satisfy 5. However, to get the subobjects for \mathbf{C} an arbitrary symmetric monoidal category is not so obvious. In this case, [2] gives definition of a *central cone*:

Definition 13 (Central Cone [2]). Let X be an object of \mathbf{C} . A *central cone* of \mathcal{T} at X is given by a pair (Z, ι) of an object Z and a morphism $\iota : Z \rightarrow \mathcal{T}X$, s.t. for any object Y , the following diagram commutes:

$$\begin{array}{ccccc} Z \otimes \mathcal{T}Y & \xrightarrow{\iota \otimes \mathcal{T}Y} & \mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau'_{X, \mathcal{T}Y}} & \mathcal{T}(X \otimes \mathcal{T}Y) \\ \iota \otimes \mathcal{T}Y \downarrow & & & & \downarrow \mathcal{T}\tau_{X, Y} \\ \mathcal{T}X \otimes \mathcal{T}Y & & & & \mathcal{T}^2(X \otimes Y) \\ \tau_{\mathcal{T}X, Y} \downarrow & & & & \downarrow \mu_{X \otimes Y} \\ \mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X, Y}} & \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & \mathcal{T}(X \otimes Y) \end{array}$$

If (Z, ι) and (Z', ι') are two central cones of \mathcal{T} at X , then a *morphism of central cones* $\varphi : (Z', \iota') \rightarrow (Z, \iota)$ is a morphism $\varphi : Z' \rightarrow Z$, such that $\iota \circ \varphi = \iota'$. Thus, central cones of \mathcal{T} at X form a category. A *terminal central cone* of \mathcal{T} at X is a central cone (Z, ι) for \mathcal{T} at X , such that for any central cone (Z', ι') of \mathcal{T} at X , there exists a unique morphism of central cones $\varphi : (Z', \iota') \rightarrow (Z, \iota)$. In other words, it is the terminal object in the category of central cones of \mathcal{T} at X .

In particular, Definition 12 gives a terminal central cone for the special case of monads on **Set**.

The authors show that central cones from [2] and central morphisms from [3] are equivalent notions by proving the proposition below:

Proposition 14. [2] Let $f : X \rightarrow \mathcal{T}Y$ be a morphism in \mathbf{C} . The pair (X, f) is a central cone of \mathcal{T} at Y iff f is central in $\mathbf{C}_{\mathcal{T}}$ in the premonoidal sense (Definition 36).

Proposition 15 (Uniqueness). If a terminal central cone for \mathcal{T} at X exists, then it is unique up to a unique isomorphism of central cones. Also, one can easily prove that if (Z, ι) is a terminal central cone, then ι is a monomorphism.

Definition 16 (Centralisable Monad [2]). We say that the monad \mathcal{T} is *centralisable* if, for any object X , a terminal central cone of \mathcal{T} at X exists. In this situation, we write $(\mathcal{Z}X, \iota_X)$ for the terminal central cone of \mathcal{T} at X .

The next theorem is one of the main theorems of [2], it shows that terminal central cones of a centralisable monad \mathcal{T} induce a commutative submonad \mathcal{Z} of \mathcal{T} .

Theorem 17 (Centre [2]). If the monad \mathcal{T} is centralisable, then the assignment $\mathcal{Z}(-)$ extends to a commutative monad $(\mathcal{Z}, \eta^{\mathcal{Z}}, \mu^{\mathcal{Z}}, \tau^{\mathcal{Z}})$ on \mathbf{C} , as the centre of \mathcal{T} . Moreover, \mathcal{Z} is a commutative submonad of \mathcal{T} and the morphisms $\iota_X : \mathcal{Z}X \rightarrow \mathcal{T}X$ constitute a monomorphism of strong monads $\iota : \mathcal{Z} \Rightarrow \mathcal{T}$ (ι from Definition 6).

The original proof in [2] uses proposition 14 which can adapt the premonoidal centre properties to immediately show that the $\eta^{\mathcal{Z}}$, $\mu^{\mathcal{Z}}$ and $\tau^{\mathcal{Z}}$ are central. Without those notions, we need to prove that they exist and are unique.

I here give the proof to this part of the theorem:

Proof of Theorem 17. This proof has three parts.

First we prove \mathcal{Z} is a functor; next we give its monad structure and prove it, by showing all of its morphisms exist and are unique (where I give my proof), and are also natural; last we prove that \mathcal{Z} is a strong and commutative monad.

First part is necessary similar as the one in [2], see C1

Second part:

Then we define its monad structure and prove its morphisms exist and are unique.

Definition of the monadic unit $\eta_X^{\mathcal{Z}}$:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X^{\mathcal{Z}}} & \mathcal{Z}X \\ \eta_X \searrow & & \swarrow \iota_X \\ & \mathcal{T}X & \end{array}$$

The original proof is very simple: $\eta_X^{\mathcal{Z}}$ is central because it is the identity morphism in $\mathcal{Z}(\mathbf{C}_T)$, thus it factors through ι_X to define $\eta_X^{\mathcal{Z}}$.

Without the central notion from [3], we need to prove that $\eta_X^{\mathcal{Z}}$ exists and is unique.

By Definition 13, the universal property of terminal central cone indicates that, for any central cone, there exists a unique morphism of central cones to \mathcal{Z} , hence we need to prove that all other arrows in this definition form central cones.

What's left is that η_X forms a central cone, it is proved by the following diagram:

$$\begin{array}{ccccc} X \otimes TY & \xrightarrow{\eta_X \otimes TY} & \mathcal{T}X \otimes TY & \xrightarrow{\tau'_{X, TY}} & \mathcal{T}(X \otimes TY) \\ & \searrow^{\tau_{X, X}} & \downarrow \eta_{X \otimes TY} & \searrow^{\tau_{X, X}} & \downarrow \mathcal{T}\tau_{X, Y} \\ & & \mathcal{T}X \otimes TY & \xrightarrow{\eta_{\mathcal{T}(X \otimes Y)}} & \mathcal{T}^2(X \otimes Y) \\ \eta_X \otimes TY \downarrow & & \downarrow \tau_{X, X} & & \downarrow \mu_{X \otimes Y} \\ \mathcal{T}X \otimes TY & \xrightarrow{\tau_{X, X}} & \mathcal{T}(X \otimes Y) & \xrightarrow{\eta_{\mathcal{T}(X \otimes Y)}} & \mathcal{T}^2(X \otimes Y) \\ \tau_{\mathcal{T}X, Y} \downarrow & \swarrow^{\mathcal{T}(\eta_X \otimes Y)} & \downarrow \mathcal{T}\eta_X & \swarrow^{\text{id}} & \downarrow \mu_{X \otimes Y} \\ \mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X, Y}} & \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & \mathcal{T}(X \otimes Y) \end{array}$$

(1) $\eta_X \otimes TY \rightarrow \mathcal{T}X \otimes TY \rightarrow \mathcal{T}(X \otimes TY)$
(2) $\eta_{X \otimes TY} \rightarrow \eta_{\mathcal{T}(X \otimes Y)}$
(3) $\eta_{\mathcal{T}(X \otimes Y)} \rightarrow \mu_{X \otimes Y}$
(4) $\mathcal{T}\eta_X \rightarrow \mu_{X \otimes Y}$
(5) $\mathcal{T}(\eta_X \otimes Y) \rightarrow \mu_{X \otimes Y}$
(6) $\eta_X \otimes TY \rightarrow \mathcal{T}X \otimes TY$

(1) η property of graded co-strength as proofed before (take normal monad as special case of graded monad); (2) η and τ are natural; (3),(4) definition of monad; (5) \mathcal{T} is a functor with η property of co-strength, and (6) τ is natural.

Next, definition of multiplication $\mu_X^{\mathcal{Z}}$:

$$\begin{array}{ccc} \mathcal{Z}^2 X & \xrightarrow{\mu_X^{\mathcal{Z}}} & \mathcal{Z}X \\ \iota_{\mathcal{Z}X} \downarrow & & \downarrow \iota_X \\ \mathcal{T}\mathcal{Z}X & \xrightarrow{\mathcal{T}\iota_X} \mathcal{T}^2 X \xrightarrow{\mu_X} & \mathcal{T}X \end{array}$$

The original proof is also simple: by definition, $\mu_X \circ \mathcal{T}\iota_X \circ \iota_{\mathcal{Z}X} = \iota_X \circ \iota_{\mathcal{Z}X}$, where $(- \circ -)$ indicates Kleisli composition. Since ι is central and Kleisli composition preserves central morphisms, it follows that this morphism factors through ι_X .

Without using the central notion from [3], we need to prove that $\mu_X^{\mathcal{Z}}$ exists and is unique.

By Definition 13, the universal property of terminal central cone indicates that, for any central cone, there exists a unique morphism of central cones to \mathcal{Z} , hence we need to prove that all other arrows in this definition form central cones.

What's left is that $\mu_X \circ \mathcal{T}\iota_X \circ \iota_{ZX}$ forms a central cone, it is proved by the following diagram:

$$\begin{array}{ccccccc}
Z^2X \otimes TY & \xrightarrow{\iota_{ZX} \otimes TY} & TZX \otimes TY & \xrightarrow{\mathcal{T}\iota_X \otimes TY} & T^2X \otimes TY & \xrightarrow{\mu_X \otimes TY} & TX \otimes TY & \xrightarrow{\tau'_{X,TY}} & \mathcal{T}(X \otimes TY) \\
\downarrow \iota_{ZX} \otimes TY & & \downarrow \tau'_{ZX,TY} & \text{(2)} & \downarrow \tau'_{TX,TY} & \text{(3)} & \downarrow \mu_{X \otimes TY} & & \downarrow \mathcal{T}\tau_{X,Y} \\
& & \mathcal{T}(ZX \otimes TY) & \xrightarrow{\mathcal{T}(\iota_X \otimes TY)} & \mathcal{T}(TX \otimes TY) & \xrightarrow{\mathcal{T}(\tau'_{X,TY})} & \mathcal{T}^2(X \otimes TY) & \text{(4)} & \\
& & \downarrow \mathcal{T}\tau_{ZX,Y} & \swarrow \mathcal{T}(\iota_X \otimes TY) & \downarrow \mathcal{T}(\tau_{TX,Y}) & & \downarrow \mathcal{T}^2\tau_{X,Y} & \text{(5)} & \\
& & \mathcal{T}^2(ZX \otimes Y) & \text{(5)} & \mathcal{T}(TX \otimes TY) & \text{(6)} & \mathcal{T}^3(X \otimes Y) & \text{(4)} & \\
& & \downarrow \mu_{ZX,Y} & \swarrow \mathcal{T}^2(\iota_X \otimes Y) & \downarrow \mu_{TX \otimes Y} & & \downarrow \mathcal{T}\mu_{X \otimes Y} & \swarrow \mu_{\mathcal{T}(X \otimes Y)} & \\
& & \mathcal{T}^2(ZX \otimes Y) & \text{(7)} & \mathcal{T}^2(TX \otimes Y) & \xrightarrow{\mathcal{T}^2\tau'_{X,Y}} & \mathcal{T}^3(X \otimes Y) & \xrightarrow{\mathcal{T}\mu_{X \otimes Y}} & \mathcal{T}^2(X \otimes Y) & \text{(13)} \\
& & \downarrow \mathcal{T}(\tau_{ZX,TY}) & \text{(8)} & \downarrow \mathcal{T}(\tau_{TX,Y}) & \text{(9)} & \downarrow \mathcal{T}(\mu_{TX \otimes Y}) & & \downarrow \mu_{\mathcal{T}(X \otimes Y)} & \\
& & \mathcal{T}^2Z X \otimes TY & \xrightarrow{\mathcal{T}\tau_{ZX,TY}} & \mathcal{T}(T^2Z X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{ZX,Y}} & \mathcal{T}^2(ZX \otimes Y) & \xrightarrow{\mu_{ZX,Y}} & \mathcal{T}(ZX \otimes Y) & \text{(10)} \\
& & \downarrow \mathcal{T}\iota_{TX,TY} & \text{(8)} & \downarrow \mathcal{T}(\iota_{TX,TY}) & \text{(9)} & \downarrow \mathcal{T}(\iota_{TX,TY}) & \text{(10)} & \downarrow \mathcal{T}(\iota_{TX,TY}) & \\
& & \mathcal{T}^2X \otimes TY & \xrightarrow{\tau_{T^2X,Y}} & \mathcal{T}(T^2X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{TX,Y}} & \mathcal{T}^2(TX \otimes Y) & \xrightarrow{\mu_{TX \otimes Y}} & \mathcal{T}(TX \otimes Y) & \text{(11)} \\
& & \downarrow \mu_{X \otimes TY} & \text{(14)} & \downarrow \mu_{X \otimes TY} & \text{(14)} & \downarrow \mu_{X \otimes TY} & \text{(11)} & \downarrow \mu_{X \otimes Y} & \\
& & TX \otimes TY & \xrightarrow{\tau_{TX,Y}} & \mathcal{T}(TX \otimes TY) & \xrightarrow{\mathcal{T}\tau'_{TX,Y}} & \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{TX \otimes Y}} & \mathcal{T}(TX \otimes Y) & \text{(12)} \\
& & \downarrow \tau_{TX,Y} & \text{(14)} & \downarrow \tau_{TX,Y} & \text{(14)} & \downarrow \tau_{TX,Y} & \text{(12)} & \downarrow \tau_{TX,Y} & \\
& & \mathcal{T}(TX \otimes Y) & \xrightarrow{\mathcal{T}\tau_{TX,Y}} & \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{TX \otimes Y}} & \mathcal{T}(TX \otimes Y) & \xrightarrow{\mu_{TX \otimes Y}} & \mathcal{T}(TX \otimes Y) & \text{(16)} \\
& & & & \downarrow \mathcal{T}^2\tau'_{X,Y} & \text{(16)} & \downarrow \mathcal{T}^2\tau'_{X,Y} & \text{(16)} & \downarrow \mathcal{T}^2\tau'_{X,Y} & \\
& & & & \mathcal{T}^3(X \otimes Y) & \xrightarrow{\mu_{\mathcal{T}(X \otimes Y)}} & \mathcal{T}^2(X \otimes Y) & \text{(17)} & \mathcal{T}^2(X \otimes Y) & \\
& & & & \downarrow \mathcal{T}\mu_{X \otimes Y} & \text{(17)} & \downarrow \mathcal{T}\mu_{X \otimes Y} & \text{(17)} & \downarrow \mathcal{T}\mu_{X \otimes Y} & \\
& & & & \mathcal{T}(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & \mathcal{T}(X \otimes Y) & \text{(17)} & \mathcal{T}(X \otimes Y) & \\
& & & & \downarrow \mathcal{T}\tau_{X,Y} & & \downarrow \mathcal{T}\tau_{X,Y} & & \downarrow \mathcal{T}\tau_{X,Y} & \\
& & & & \mathcal{T}(TX \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X,Y}} & \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & \mathcal{T}(X \otimes Y) & \\
& & & & & & \downarrow \mu_{X \otimes Y} & & \downarrow \mu_{X \otimes Y} & \\
& & & & & & \mathcal{T}(X \otimes Y) & & \mathcal{T}(X \otimes Y) & \\
& & & & & & \downarrow \mu_{X \otimes Y} & & \downarrow \mu_{X \otimes Y} & \\
& & & & & & \mathcal{T}(X \otimes Y) & & \mathcal{T}(X \otimes Y) &
\end{array}$$

- (1) ι_{ZX} is central; (2) τ' and ι_X are natural; (3) proposition 9, μ property of graded co-strength (take normal monad as special case of graded monad); (4) μ and τ are natural; (5) \mathcal{T} is a functor and τ, ι_X are natural; (6) \mathcal{T} is a functor and ι_X is central; (7) μ and ι_X are natural; (8) τ and ι_X are natural; (9) \mathcal{T} is a functor and τ', ι_X are natural; (10) μ and ι_X are natural; (11) μ and τ' are natural; (12),(13) and (17) definition of monad; (14) μ and τ are natural; (15) \mathcal{T} is a functor and proposition 9, μ property of graded co-strength, and (16) μ and τ' are natural;

Last, definition of the strength $\tau_{X,Y}^Z$:

$$\begin{array}{ccc}
W \otimes ZX & \xrightarrow{\tau_{W,X}^Z} & \mathcal{Z}(W \otimes X) \\
W \otimes \iota_X \downarrow & & \downarrow \iota_{W \otimes X} \\
W \otimes TX & \xrightarrow{\tau_{W,X}} & \mathcal{T}(W \otimes X)
\end{array}$$

The original proof is that, by definition, $\tau_{A,B} \circ (A \otimes \iota_B) = A \otimes_r \iota_B$. Central morphisms are preserved by the premonoidal products.

Without using the central notion from [3], we need to prove that μ_X^Z exists and is unique.

By Definition 13, the universal property of terminal central cone indicates that, for any central cone, there exists a unique morphism of central cones to \mathcal{Z} , hence we need to prove that all other arrows in this definition form central cones.

What's left is that $\tau_{W,X} \circ (W \otimes \iota_X)$ forms a central cone, it is proved by the following diagram:

The diagram is a large commutative diagram with 14 numbered nodes (1) through (14) and various arrows representing natural transformations and functors. The nodes are arranged in a grid-like structure. The top row starts with $(W \otimes ZX) \otimes TY$ and ends with $T((W \otimes X) \otimes TY)$. The left column starts with $(W \otimes \iota_X) \otimes TY$ and ends with $T(T(W \otimes X) \otimes Y)$. The right column starts with $T((W \otimes X) \otimes TY)$ and ends with $T((W \otimes X) \otimes Y)$. The bottom row starts with $T(T(W \otimes X) \otimes Y)$ and ends with $T((W \otimes X) \otimes Y)$. The diagram includes arrows labeled with α , τ , T , μ , and τ' , along with their respective indices and natural transformations. The nodes are connected by a series of horizontal, vertical, and diagonal arrows, forming a complex web of relationships that demonstrate the central cone property.

(1), (3) α , ι_X are natural; (2) lemma 34; (4) ι_X is central; (5), (7) τ is natural; (6) T is a functor, $\alpha \circ \alpha^{-1} = id$, definition on strength; (8), (10), (11) definition of strength; (9) τ , τ' are natural; (12), (14) μ and α^{-1} are natural; (13) T is a functor, lemma 34.

The last three definitions are exactly those of a morphism of strong monads (see Definition 6).

The rest of the proof is also necessarily similar as that of [2], see C2. \square

This theorem shows that centralisable monads always induce a canonical commutative submonad.

Furthermore, [2] introduces the notion of centralisable submonad, which is a commutative submonad of a centralisable monad. In this work we extend it in graded monads in VIII, first we need to construct the centre notion on (strong) graded monad.

VI. THE CENTRE OF A STRONG GRADED MONAD

As already seen in III, given a category \mathbf{C} , a monad is an endofunctor $\mathcal{T} : \mathbf{C} \rightarrow \mathbf{C}$, meanwhile a monoid \mathcal{G} -graded monad on \mathbf{C} is a lax monoidal functor as a discrete monoidal category from the monoid \mathcal{G} to endofunctor category of $\mathbf{C} : (\mathcal{G}, i, *) \rightarrow ([\mathbf{C}, \mathbf{C}], Id_{\mathbf{C}}, \cdot)$.

We can get the intuition that the construction of the centre of a strong graded monad would be involving the constructions not only on the lax monoidal functor, which is the monadic structure, but also on the gradation \mathcal{G} , and the relation between them, by using the morphism between monoid graded strong monads defined in Definition 11.

Let's revisit the centre of a monoid, the centre is defined as the set of all those elements that commute with all other elements:

Definition 18 (Centre of a monoid). Given a monoid $\mathcal{G} : (G, i, *)$, we say the centre of the monoid, written $\mathcal{Z}(\mathcal{G})$, is

$$\mathcal{Z}(\mathcal{G}) \stackrel{\text{def}}{=} \{a \in G \mid \forall b \in G, a * b = b * a\}$$

Moreover, $\mathcal{Z}(\mathcal{G})$ is a submonoid of \mathcal{G} which $G \subseteq G'$.

Remark 19 (Remark on Gradations of the Centre). Throughout the rest of this report, we consider the most common case: the gradation of the centre of the graded monad being the centre of the original gradation. Let monoid $\mathcal{G} : (G, i, *)$ be the gradation of the centre, and $\mathcal{G}' : (G', i', \otimes)$ be that of the original. G is the subset of G' , now the homomorphism ϕ defined in Definition 11 between gradations becomes subset inclusion $G \subseteq G'$, $i, *$ are the same as i', \otimes , and ϕa is just a .²

Next we give the centre on strong graded monads on **Set**.

Definition 20 (Graded Centre in **Set**). Let monoid $\mathcal{G} : (G, i, *)$ be the centre of monoid $\mathcal{G}' : (G', i', \otimes)$, and a strong \mathcal{G}' -graded monad $(\mathcal{T}, \eta, \mu, \tau)$ on **Set** with a family of right strength τ' indexed by elements of \mathcal{G}' .

²However, in this report, we keep using i', \otimes and ϕa even though they are actually the same as $i, *$ and a , for better tracking of the origin of the gradations, also for convenience of further more complicated cases, e.g. the \mathcal{G} could also be a monoid which is homomorphic to some submonoid of \mathcal{G}' .

Take an arbitrary $a \in G$, we say that the \mathcal{G} -graded *centre* of \mathcal{T} at (a, X) , written $\overset{a}{\mathcal{Z}}X$, is the set

$$\overset{a}{\mathcal{Z}}X \stackrel{\text{def}}{=} \{t \in \overset{\phi a}{\mathcal{T}}X \mid \forall b' \in G', \forall Y \in \text{Ob}(\mathbf{Set}), \forall s \in \overset{\phi a}{\mathcal{T}}Y, \mu(\overset{\phi a}{\mathcal{T}}\tau'(\tau(t, s))) = \mu(\overset{b'}{\mathcal{T}}\tau(\tau'(t, s)))\}.$$

We write $\overset{a}{\iota}_X : \overset{a}{\mathcal{Z}}X \subseteq \overset{\phi a}{\mathcal{T}}X$ for the indicated subset inclusion.³

Now the centre of a strong \mathcal{G}' -graded monad \mathcal{T} on \mathbf{Set} at X is a family of subsets of \mathcal{G}' -graded $\mathcal{T}X$ indexed by elements of \mathcal{G} , which contains all monadic elements for which Definition 10 holds when X is fixed, for all elements of G' and Y ranges over all sets.

For the rest of this section, we assume \mathbf{C} is a symmetric monoidal category, V, W, X, Y are objects in \mathbf{C} ; monoid $\mathcal{G} : (G, i, *)$ be the centre of monoid $\mathcal{G}' : (G', i', \otimes)$, \mathcal{T} is graded by monoid \mathcal{G}' over \mathbf{C} ; a, b, c are elements in \mathcal{G} and a', b', c' are elements in \mathcal{G}' .

To extend it from \mathbf{Set} to categories, we introduce the graded central cone:

Definition 21 (Graded Central Cone). Let X be an object of \mathbf{C} and $a \in G$. A *graded central cone* of \mathcal{G}' -graded \mathcal{T} at (a, X) , is given by a pair (Z, ι) of object Z and a morphism $\iota : Z \rightarrow \overset{\phi a}{\mathcal{T}}X$, s.t. for any object Y in \mathbf{C} and any $b' \in G'$, the following diagram commutes:

$$\begin{array}{ccccc} Z \otimes \overset{b'}{\mathcal{T}}Y & \xrightarrow{\iota \otimes \overset{b'}{\mathcal{T}}Y} & \overset{\phi a}{\mathcal{T}}X \otimes \overset{b'}{\mathcal{T}}Y & \xrightarrow{\tau'_{X, \overset{b'}{\mathcal{T}}Y}} & \overset{\phi a}{\mathcal{T}}(X \otimes \overset{b'}{\mathcal{T}}Y) \\ \downarrow \iota \otimes \overset{b'}{\mathcal{T}}Y & & & & \downarrow \overset{\phi a}{\mathcal{T}}\tau_{X, Y} \\ \overset{\phi a}{\mathcal{T}}X \otimes \overset{b'}{\mathcal{T}}Y & & & & \overset{\phi a}{\mathcal{T}} \cdot \overset{b'}{\mathcal{T}}(X \otimes Y) \\ \downarrow \tau_{\overset{\phi a}{\mathcal{T}}X, Y} & & & & \downarrow \mu_{X \otimes Y}^{\phi a, b'} \\ \overset{b'}{\mathcal{T}}(\overset{\phi a}{\mathcal{T}}X \otimes Y) & \xrightarrow{\overset{b'}{\mathcal{T}}\tau'_{X, Y}} & \overset{b'}{\mathcal{T}} \cdot \overset{\phi a}{\mathcal{T}}(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}^{b', \phi a}} & \overset{b' \otimes \phi a}{\mathcal{T}}(X \otimes Y) \\ & & & & \parallel \\ & & & & \overset{\phi a \otimes b'}{\mathcal{T}}(X \otimes Y) \end{array}$$

Given (Z', ι') and (Z, ι) two graded central cones of \mathcal{G}' -graded \mathcal{T} at (a, X) , a *morphism of graded central cones* $\varphi : (Z', \iota') \rightarrow (Z, \iota)$ is a morphism s.t. $\varphi : Z' \rightarrow Z$, s.t. $\iota \circ \varphi = \iota'$. Thus, graded central cones of \mathcal{T} at (a, X) form a category. A *terminal graded central cone* of \mathcal{G}' -graded \mathcal{T} at (a, X) is a central cone (Z, ι) for \mathcal{T} at X , s.t. for any graded central cone (Z', ι') of \mathcal{T} at (a, X) , there exists a unique morphism of graded central cones $\varphi : (Z', \iota') \rightarrow (Z, \iota)$. In other words, it is the terminal object in the category of graded central cones of \mathcal{T} at (a, X) .

In particular, Definition 20 gives a terminal graded central cone for the special case of graded monads on \mathbf{Set} .

The central cone

Proposition 22 (Uniqueness). *If a terminal graded central cone for \mathcal{G}' -graded \mathcal{T} at (a, X) exists, then it is unique up to a unique isomorphism of graded central cones. Also, one can easily prove that if (Z, ι) is a terminal graded central cone, then ι is a monomorphism.*

Proof. Similar to normal version, omitted. □

Definition 23 (Centralisable Graded Monad). We say that the \mathcal{G}' -graded monad \mathcal{T} over \mathbf{C} is *centralisable* if, for any object X in \mathbf{C} , for any element a in $\mathcal{Z}(\mathcal{G}')$ (which is \mathcal{G}), a terminal graded central cone of \mathcal{T} at (a, X) exists. In this situation, we write $(\overset{a}{\mathcal{Z}}X, \overset{a}{\iota}_X)$ for the terminal graded central cone of \mathcal{T} at (a, X) .

For a centralisable \mathcal{G}' -graded monad \mathcal{T} , the next theorem shows that its terminal graded central cones induce a commutative $\mathcal{Z}(\mathcal{G}')$ (which is \mathcal{G})-graded submonad \mathcal{Z} of \mathcal{T} . Which we now call the centre of the graded monad. And its proof reveals constructively how the graded strong monad structure arises from them.

³Note that we write b' here for element in \mathcal{G}' , elements with $'$ indicate they are from original monoid, and those without are from its centre, different color to each letter is just for convenience.

Theorem 24 (Centre). *If the \mathcal{G}' -graded monad \mathcal{T} is centralisable, then the assignment $\mathcal{Z}(-)$ extends to a commutative \mathcal{G} -graded monad $(\mathcal{Z}, \eta^{\mathcal{Z}}, \mu^{\mathcal{Z}}, \tau^{\mathcal{Z}})$ on \mathbf{C} , as the centre of \mathcal{T} . Moreover, \mathcal{Z} is a commutative \mathcal{G} -graded submonad of \mathcal{T} and the family of morphisms $\iota_X^{\mathcal{Z}} : \mathcal{Z}X \rightarrow \mathcal{T}X$, constitute of a monomorphism between strong graded monads $\mathcal{Z} \Rightarrow \mathcal{T}$ indexed by elements in \mathcal{G} , in the sense of Definition 11.*

Proof of Theorem 24. This proof has 3 parts.

First we prove that \mathcal{Z} is a functor; then we give its graded monad structure and prove it, by showing all of its morphisms exist and are unique, and also natural; last we show that it's a strong and commutative graded monad.

First part:

The first part is following same proof strategy as that of Theorem 17 in [2], but on graded monads, which are lax monoidal functors between gradation and endofunctor category of \mathbf{C} .

Recall that \mathcal{Z} maps every object X to its terminal central cone at (a, X) . Let $f : X \rightarrow Y$ be a morphism. $\mathcal{T}f \circ \iota_X^{\mathcal{Z}} : \mathcal{Z}X \rightarrow \mathcal{T}Y$ is a central cone according to Lemma 32. Therefore, by proposition 22, we can define $\mathcal{Z}f$ as the unique map such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Z}X & \xrightarrow{\mathcal{Z}f} & \mathcal{Z}Y \\ \iota_X \downarrow & & \downarrow \iota_Y \\ \mathcal{Z}X & \xrightarrow{\mathcal{Z}f} & \mathcal{Z}Y \end{array}$$

It follows directly that \mathcal{Z} maps the identity to the identity, and that ι is natural. \mathcal{Z} also preserves composition, which follows by the commutative diagram below:

$$\begin{array}{ccc} \mathcal{Z}W & \xrightarrow{\mathcal{Z}g} & \mathcal{Z}X \\ \mathcal{Z}g \downarrow & & \downarrow \mathcal{Z}f \\ \mathcal{Z}X & \xrightarrow{\mathcal{Z}f} & \mathcal{Z}Y \\ \mathcal{Z}f \downarrow & & \downarrow \mathcal{Z}f \\ \mathcal{Z}Y & \xrightarrow{\mathcal{Z}f} & \mathcal{Z}Y \end{array}$$

$\mathcal{Z}(f \circ g)$ (dashed arrow from $\mathcal{Z}W$ to $\mathcal{Z}Y$)
 $\mathcal{T}(f \circ g)$ (solid arrow from $\mathcal{T}W$ to $\mathcal{T}Y$)

This proves that \mathcal{Z} is a functor.

Second part.

Then we describe its graded monad structure and prove all its morphisms exist and are unique.

This part I give my own proof following same proof strategy I used in the proof of Theorem 24 but on graded monads.

Definition of the monadic unit $\eta_X^{\mathcal{Z}}$:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X^{\mathcal{Z}}} & \mathcal{Z}X \\ \eta_X \searrow & & \downarrow \iota_X \\ \mathcal{T}X & \xrightarrow{\phi^i} & \mathcal{T}X \end{array}$$

By Definition 21, the universal property of terminal graded central cone indicates that, for any graded central cone at (a, X) , there exists a unique morphism of graded central cones to \mathcal{Z} , hence we need to prove that all other arrows in this definition form graded central cones.

What's left is that $\eta_X^{\mathcal{Z}}$ forms a central cone, it is proved by the following diagram:

η_X is central by the following commutative diagram:

The diagram illustrates the naturality and compatibility of the natural transformations η and τ with the monad multiplication μ and the functor \mathcal{T} . The nodes represent various tensor products and their images under \mathcal{T} . The arrows represent the natural transformations and their compositions. The properties (1) through (6) are:

- (1) $\eta_X \otimes \mathcal{T}Y \xrightarrow{\eta_X \otimes \mathcal{T}Y} \mathcal{T}X \otimes \mathcal{T}Y \xrightarrow{\eta_{X \otimes \mathcal{T}Y}} \mathcal{T}(X \otimes \mathcal{T}Y)$
- (2) $\eta_{X \otimes \mathcal{T}Y} \xrightarrow{\eta_{X \otimes \mathcal{T}Y}} \mathcal{T}(X \otimes \mathcal{T}Y)$
- (3) $\mathcal{T}(X \otimes \mathcal{T}Y) \xrightarrow{\eta_{\mathcal{T}(X \otimes Y)}} \mathcal{T} \cdot \mathcal{T}(X \otimes Y)$
- (4) $\mathcal{T}(X \otimes Y) \xrightarrow{\eta_{X \otimes Y}} \mathcal{T}(X \otimes Y)$
- (5) $\mathcal{T}(\eta_X \otimes Y) \xrightarrow{\mathcal{T}(\eta_X \otimes Y)} \mathcal{T} \cdot \mathcal{T}(X \otimes Y)$
- (6) $\tau_{X,Y} \xrightarrow{\tau_{X,Y}} \mathcal{T}\tau_{X,Y}$

(1) η property of graded co-strength; (2) η and τ are natural; (3),(4) definition of graded monad; (5) \mathcal{T} is a functor with property of graded co-strength; (6) τ is natural and rest are equalities.

Next, definition of multiplication $\mu_X^{\mathcal{Z}}$:

$$\begin{array}{ccc}
 \begin{array}{c} \overset{a}{\mathcal{Z}} \cdot \overset{b}{\mathcal{Z}}X \\ \downarrow \iota_b \\ \overset{\phi a}{\mathcal{T}} \cdot \overset{b}{\mathcal{Z}}X \end{array} & \xrightarrow{\mu_X^{\mathcal{Z}}} & \begin{array}{c} \overset{a*b}{\mathcal{Z}}X \\ \downarrow \iota_X \\ \overset{\phi(a*b)}{\mathcal{T}}X \end{array} \\
 \downarrow \iota_b & & \downarrow \iota_X \\
 \begin{array}{c} \overset{\phi a}{\mathcal{T}} \cdot \overset{b}{\mathcal{Z}}X \\ \xrightarrow{\phi a} \overset{\phi a}{\mathcal{T}} \cdot \overset{\phi b}{\mathcal{T}}X \\ \xrightarrow{\mu_X^{\mathcal{T}}} \overset{\phi a \otimes \phi b}{\mathcal{T}}X \\ \equiv \overset{\phi(a*b)}{\mathcal{T}}X \end{array} & &
 \end{array}$$

By Definition 21, the universal property of terminal graded central cone indicates that, for any graded central cone at (a, X) , there exists a unique morphism of graded central cones to $\overset{a}{\mathcal{Z}}$, hence we need to prove that all other arrows in this definition form graded central cones.

What's left is that $\mu_X^{\mathcal{T}} \circ \mathcal{T}\iota_X \circ \iota_b$ forms a central cone, it is proved by the following diagram:

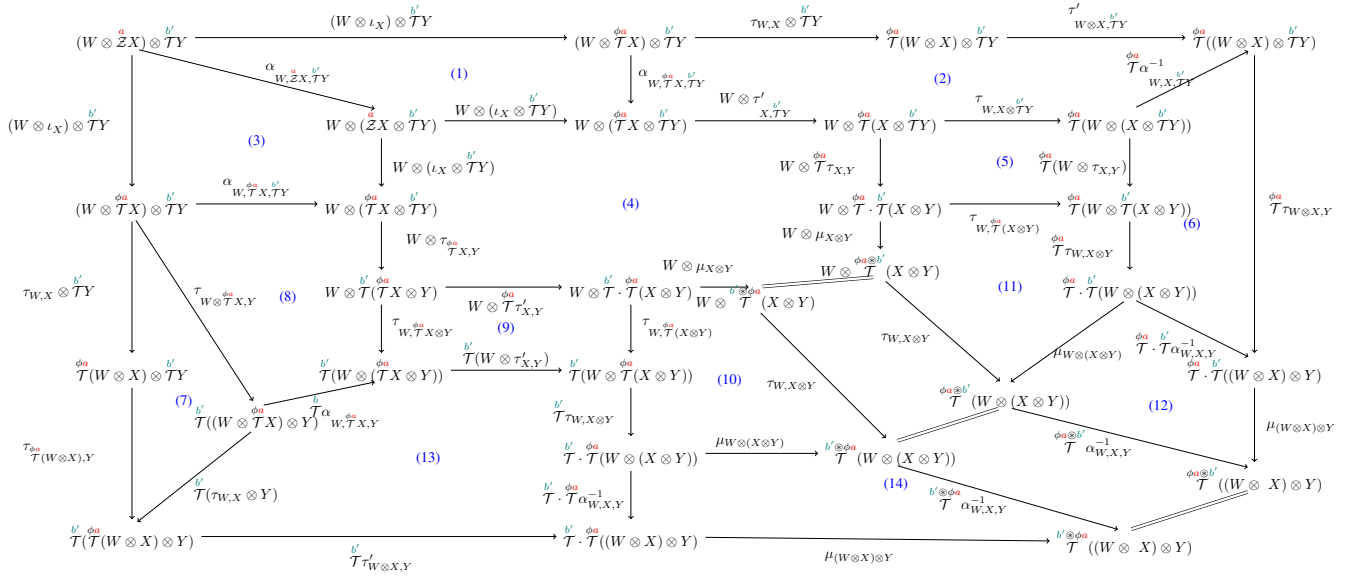
(1) ι_{ZX} is central; (2) τ' and ι_X are natural; (3) proposition 9, μ property of graded co-strength (take normal monad as special case of graded monad); (4) μ and τ are natural; (5) \mathcal{T} is a functor and τ, ι_X are natural; (6) \mathcal{T} is a functor and ι_X is central; (7) μ and ι_X are natural; (8) τ and ι_X are natural; (9) \mathcal{T} is a functor and τ', ι_X are natural; (10) μ and ι_X are natural; (11) μ and τ' are natural; (12),(13) and (17) definition of monad; (14) μ and τ are natural; (15) \mathcal{T} is a functor and proposition 9, μ property of graded co-strength; (16) μ and τ' are natural and rest are equalities.

Last, definition of the strength $\tau_{W,X}^Z$:

$$\begin{array}{ccc}
 W \otimes ZX & \xrightarrow{\tau_{W,X}^Z} & Z(W \otimes X) \\
 W \otimes \iota_X \downarrow & & \downarrow \iota_{W \otimes X} \\
 W \otimes \mathcal{T}X & \xrightarrow{\tau_{W,X}^a} & \mathcal{T}(W \otimes X)
 \end{array}$$

By Definition 21, the universal property of terminal graded central cone indicates that, for any graded central cone at (a, X) , there exists a unique morphism of graded central cones to Z , hence we need to prove that all other arrows in this definition form graded central cones.

What's left is that $\tau_{W,X} \circ (W \otimes \iota_X)$ forms a central cone, it is proved by the following diagram:

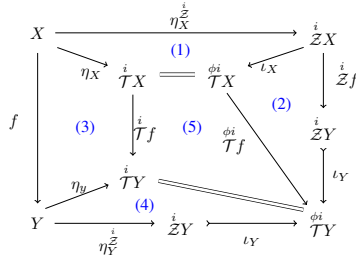


(1), (3) α , ι_X are natural; (2) lemma 34; (4) ι_X is central; (5), (7) τ is natural; (6) T is a functor, $\alpha \circ \alpha^{-1} = id$, definition on graded strength; (8), (10), (11) definition of strength; (9) τ , τ' are natural; (12), (14) μ and α^{-1} are natural; (13) T is a functor, lemma 34; and rest are equalities.

The rest of the proof is following the same strategy as that of Theorem 17 in [2].

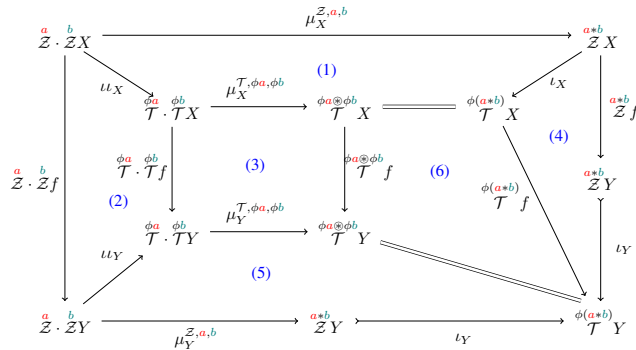
The last three definitions are exactly those of a morphism of strong graded monads (see Definition 11).

Using the fact that ι is monic (see Lemma 33), the following commutative diagram shows that η^Z is natural:



(1), (4) definition of η^Z ; (2) ι is natural; (3) η is natural; and (5) equality. Thus, we have proven that for any $f : X \rightarrow Y$, $\iota_Y \circ \mathcal{Z}f \circ \eta_X^Z = \iota_Y \circ \eta_Y^Z \circ f$. Besides, ι is monic, thus $\mathcal{Z}f \circ \eta_X^Z = \eta_Y^Z \circ f$ which proves that η^Z is natural. We will prove all the remaining diagrams with the same reasoning.

The following commutative diagram shows that μ^Z is natural.

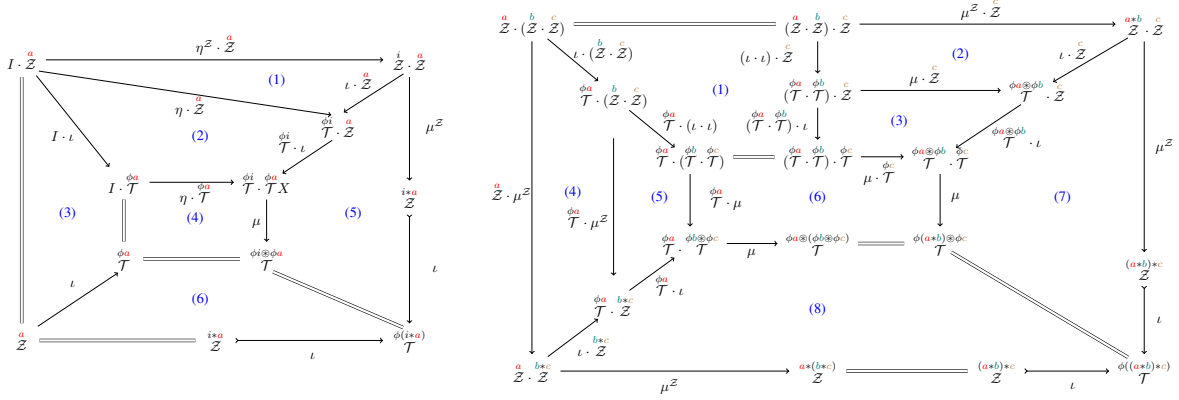


(1) (5) definition of μ^Z ; (2) (4) ι is natural; (3) μ is natural and (6) equality.

The commutative diagrams showing that τ^Z is natural are just as those on normal monad by replacing \mathcal{Z} as $\overset{a}{\mathcal{Z}}$, \mathcal{T} as $\overset{\phi a}{\mathcal{T}}$ and

ι as family of morphisms $\iota^a : \mathcal{Z} \rightarrow \mathcal{T}$ indexed by elements in \mathcal{G} .

The following commutative diagrams prove that \mathcal{Z} is a graded monad ($\mathcal{Z} \cdot I$ is omitted because it is very similar to $I \cdot \mathcal{Z}$).



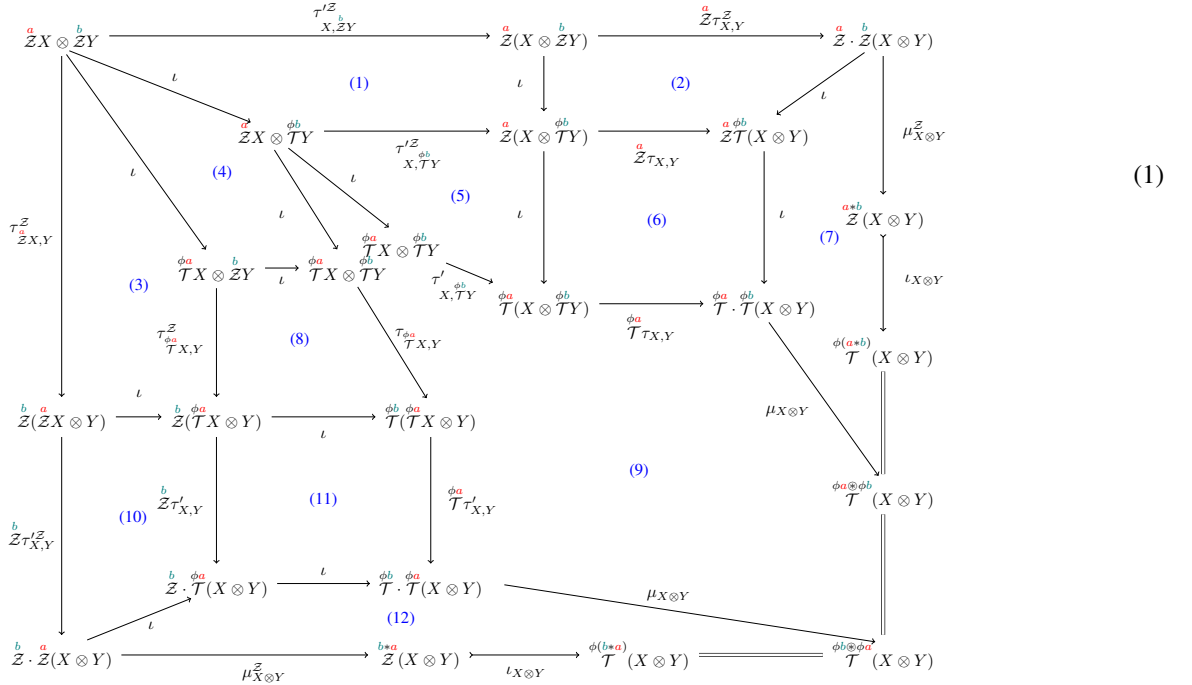
Left : (1) definition of $\mu^{\mathcal{Z}}$ and \mathcal{Z} is natural; (2) ι is natural, (3) (6) equality; (4) definition of η and (5) definition of $\mu^{\mathcal{Z}}$.

Right : (1) equality; (2) definition of $\mu^{\mathcal{Z}}$, \mathcal{Z} is a functor; (3) μ and ι are functors; (4) $\mu^{\mathcal{Z}}$ and ι are functors; (5) (7) (8) definition of $\mu^{\mathcal{Z}}$, \mathcal{T} is a functor and (6) definition of graded monad.

Last Part:

\mathcal{Z} is proven strong with very similar diagrams.

The following commutative diagram proves that \mathcal{Z} is a commutative graded monad:



(1) $\tau^{I\mathcal{Z}}$ is natural; (2) definition of $\tau^{\mathcal{Z}}$; (3) $\tau^{\mathcal{Z}}$ is natural; (4) \mathbf{C} is monoidal; (5) definition of $\tau^{I\mathcal{Z}}$; (6) ι is natural; (7) definition of $\mu^{\mathcal{Z}}$; (8) definition of $\tau^{\mathcal{Z}}$; (9) ι is central; (10) definition of $\tau^{I\mathcal{Z}}$; (11) ι is natural and (12) definition of $\mu^{\mathcal{Z}}$. \square

This theorem shows that centralisable graded monads always induce a canonical commutative graded submonad.

VII. EXAMPLES OF CENTRES OF STRONG GRADED MONADS

In \mathbf{Set} , the terminal central graded cones are defined by taking appropriate subsets. Meanwhile, centre of a monoid always exists, we can conclude that every (strong) graded monad on \mathbf{Set} is centralisable. In fact, it also holds for many other naturally occurring categories which the objects of the category have a suitable notion of subobject (e.g., subsets in \mathbf{Set} , subspaces in \mathbf{Vect}).

A. Centralisable Graded Monads

First we give an example of centralisable graded monads in **Set**.

Example 25. Multi-error graded writer monad is a writer monad which stops the computation whenever it meets the first error.

The (normal) writer monad is induced by a monoid (M, e, m) , is $\mathcal{T} = (- \times M) : \mathbf{Set} \rightarrow \mathbf{Set}$. The centre \mathcal{Z} of \mathcal{T} is given by the commutative monad $(- \times Z(M)) : \mathbf{Set} \rightarrow \mathbf{Set}$, where $Z(M)$ is the centre of the monoid M and where the monad data is given by the (co)restrictions of the monad data of \mathcal{T} . \mathcal{T} is commutative iff M is commutative.

It is now graded by a monoid $(E, t, *)$ of possible outcomes, t indicates the result is not an error, e_a, e_b are arbitrary elements in the set of errors except for the first one, e_1 being the first error.

The $*$ composition is defined as:

*	t	e_1	e_a	e_b
t	t	e_1	e_a	e_b
e_1	e_1	e_1	e_1	e_1
e_a	e_a	e_1	e_a	e_b
e_b	e_b	e_1	e_a	e_b

It is not a commutative monoid because we have the rules $e_a * e_b = e_b$ and $e_b * e_a = e_a$, means that the computation stops at the earlier error it meets.

The centre of this monoid is $(\{t, e_1\}, t, *)$, and the centre of this graded writer monad is the commutative monad $(- \times Z(M)) : \mathbf{Set} \rightarrow \mathbf{Set}$ graded only by those two elements, it is a commutative $(\{t, e_1\}, t, *)$ -graded submonad.

For examples which the subobject can be easily defined:

Example 26. Every strong monoid graded monad on the category **Top** (whose objects are topological spaces and the morphisms are continuous maps between them) is centralisable.

We can write $\overset{a}{\mathcal{T}} : \mathbf{Top} \rightarrow \mathbf{Top}$ for an arbitrary strong \mathcal{G} monoid graded monad on **Top**, the terminal graded central cone of \mathcal{T} at (a, X) , which $a \in \mathcal{Z}(\mathcal{G})$, is given by the space $\overset{a}{\mathcal{Z}}X \subseteq \overset{a}{\mathcal{T}}X$ indexed by elements of $\mathcal{Z}(\mathcal{G})$, is the set

$$\overset{a}{\mathcal{Z}}X \stackrel{\text{def}}{=} \left\{ t \in \overset{a}{\mathcal{T}}X \mid \forall b \in \mathcal{G}, \forall Y \in \text{Ob}(\mathbf{Top}), \forall s \in \overset{a}{\mathcal{T}}Y. \mu(\overset{a}{\mathcal{T}}\tau'(\tau(t, s))) = \mu(\overset{b}{\mathcal{T}}\tau'(\tau(t, s))) \right\}$$

and whose topology is the subspace topology inherited from $\overset{a}{\mathcal{T}}X$ at (a, X) .

Example 27. Every strong monoid \mathcal{G} -graded monad on the category **Vect** is centralisable. One simply defines the $\mathcal{Z}(\mathcal{G})$ indexed subset $\mathcal{Z}X$ and construct a graded central cone similarly.

Due to the limited space, we will refrain from providing an exhaustive list of every centralisable example. However, it is worth noting that nearly all encountered cases thus far have proven to be centralisable. Nonetheless, it is important to emphasize that not every graded strong monad possesses the property of being centralisable.

B. A Non-centralisable Monad

A monad might not be centralisable because in some cases the notion of subobject can be tricky – not every subset of a given set is an object of the category. An example of non-centralisable monad found in [2]: consider the writer monad induced by Dihedral group \mathbb{D}_4 , it is actually the only case we know so far.

VIII. CENTRAL GRADED SUBMONADS

Now we introduce the *central graded submonads* of a strong graded monad. Central graded submonads are more general compared to the centre. In fact, the centre of a strong graded monad, whenever it exists, is the largest central graded submonad.

Theorem 28 (Centrality). *Let monoid $\mathcal{G} : (G, i, *)$ be the centre of monoid $\mathcal{G}' : (G', i', \otimes)$. Let \mathbf{C} be a symmetric monoidal category and \mathcal{T} a strong \mathcal{G}' -graded monad on it. Let \mathcal{S} be a strong \mathcal{G} -graded submonad of \mathcal{T} with a morphism $\iota : \mathcal{S} \Rightarrow \mathcal{T}$, the strong submonad monomorphism. The following are equivalent:*

- 1) For any object a in \mathcal{G} , any object X of \mathbf{C} , $(\overset{a}{\mathcal{S}}X, \iota_X)$ is a graded central cone for \mathcal{T} at (a, X) ;
- 2) \mathcal{S} is a commutative graded submonad of the centre of \mathcal{T} .

Proof. $(1 \Rightarrow 2)$: Each $\iota_X^{\mathcal{S}} : \overset{a}{\mathcal{S}}X \Rightarrow \overset{a}{\mathcal{T}}X$ factorizes through the terminal central cone $\iota_X^{\mathcal{Z}}$. A strong graded monad morphism $\mathcal{S} \Rightarrow \mathcal{Z}$ arises from those factorizations.

$(2 \Rightarrow 1)$: Let us write \mathcal{Z} the centre of \mathcal{T} , $\iota^{\mathcal{S}} : \mathcal{S} \Rightarrow \mathcal{Z}$ and $\iota^{\mathcal{Z}} : \mathcal{Z} \Rightarrow \mathcal{T}$ the submonad morphisms. The components of $\iota^{\mathcal{Z}}$ are terminal central cones, hence exist and are unique, so $\iota^{\mathcal{Z}} \circ \iota^{\mathcal{S}}$ also exists and is unique by Lemma 31. Thus, the components of the submonad morphism from \mathcal{S} to \mathcal{T} exist and are unique. \square

Definition 29 (Central Graded Submonad). Given a strong graded submonad \mathcal{S} of \mathcal{T} , we say that \mathcal{S} is a *central graded submonad* of \mathcal{T} if it satisfies any one of the above equivalent criteria from Theorem 28.

Just like the centre of a strong graded monad, any central graded submonad also is commutative and the above theorem shows that central graded submonads have a similar structure to the centre of a strong graded monad. The final statement shows that we may see the centre (whenever it exists) as the largest central graded submonad of \mathcal{T} . The centre of a strong graded monad often does exist, so the last criterion also provides a simple way to determine whether a graded submonad is central or not.

IX. CONCLUSION AND FUTURE WORK

In this work, we successfully constructed the centre of graded monads, and central graded submonads, which are graded by monoids over symmetric monoidal categories. From the standpoint of category theory, our work illustrates that the concept of a centre can be established on a lax monoidal functor between different monoidal structures. Moreover, in a computational context, we extend the centre construction to monads wherein effects are explicitly indexed by monoids.

Moving forward, a pressing next step is to extend these results on pomonoid-graded monads, then we can give computational meaning base on the pomonoid structure. Another urgent next step, as the reader might notice, involves addressing the gaps in definitions during the construction of the centre, effectively connecting our constructions with existing works. For instance, we should investigate whether the \mathcal{Z} in Definition 23 actually a \mathcal{G} -grading object defined in [8]?

As elucidated in Definition 11, we can have a morphism between strong graded monads if there's a homomorphism between distinct monoids for gradations. As indicated in Remark 19, this study selects the gradation monoid to be the subset of original monoid. Hence, in order to fully harness the potential of Definition 11, we can explore scenarios on different choices of gradations, s.t. there exist possibilities of building such a homomorphism between them. Such as different category-graded monads, etc.

Further exploration could involve seeking a more generalized approach for constructing commutative structures and centres. The gradation of a commutative graded monad, is not necessarily commutative if we only want the diagram in Definition 10 to commute. Consider such a counter example: a monad graded by a string concatenation monoid $(S, ++, \text{""})$, which is a well-known non-commutative monoid. If we define the graded monad to always return same object as it takes, then it's commutative with whatever gradings. Hence, one compelling challenge for future work could be to determine an "if and only if" condition for the commutativity of graded monads.

The generality also extends to the choice of the base category and gradations. For instance, the monads graded by pomonoids over skew monoidal categories as introduced in [8], or even biased monoidal categories. A hard question for future work would be: is there a minimum monoidal requirement for gradations and the base categories, s.t. we can construct the centre ?

In the application perspective, Central Submonad Calculus is introduced in [2], offering a computational interpretation around the notion center of monads. It would be intriguing to further develop this line of thought, extending it to build computational meaning pertaining to explicitly tracked effects and coeffects, following the principles outlined in [1].

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REFERENCES

- [1] M. Gaboardi, S. Katsumata, D. A. Orchard, F. Breuvert, and T. Uustalu, “Combining effects and coefficients via grading,” in *Proceedings of the 21st ACM SIGPLAN International Conference on Functional Programming, ICFP 2016, Nara, Japan, September 18-22, 2016*, J. Garrigue, G. Keller, and E. Sumii, Eds. ACM, 2016, pp. 476–489. [Online]. Available: <https://doi.org/10.1145/2951913.2951939>
- [2] T. Carette, L. Lemonnier, and V. Zamdzhiev, “Central Submonads and Notions of Computation,” 2022, working paper or preprint. [Online]. Available: <https://arxiv.org/abs/2207.09190>
- [3] J. Power and E. P. Robinson, “Premonoidal categories and notions of computation,” *Math. Struct. Comput. Sci.*, vol. 7, pp. 453–468, 1997.
- [4] S. Fujii, S. Katsumata, and P. Mellès, “Towards a formal theory of graded monads,” in *Foundations of Software Science and Computation Structures - 19th International Conference, FOSSACS 2016, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings*, ser. Lecture Notes in Computer Science, B. Jacobs and C. Löding, Eds., vol. 9634. Springer, 2016, pp. 513–530. [Online]. Available: https://doi.org/10.1007/978-3-662-49630-5_30
- [5] S. M. Lane, *Categories for the Working Mathematician*. Springer, 1971.
- [6] E. Moggi, “Computational lambda-calculus and monads,” in *Proceedings of the Fourth Annual Symposium on Logic in Computer Science (LICS '89), Pacific Grove, California, USA, June 5-8, 1989*. IEEE Computer Society, 1989, pp. 14–23. [Online]. Available: <https://doi.org/10.1109/LICS.1989.39155>
- [7] —, “Notions of computation and monads,” *Inf. Comput.*, vol. 93, no. 1, pp. 55–92, 1991. [Online]. Available: [https://doi.org/10.1016/0890-5401\(91\)90052-4](https://doi.org/10.1016/0890-5401(91)90052-4)
- [8] F. Breuvert, D. McDermott, and T. Uustalu, “Caonical gradings of monads,” *EPTCS,ACT2022*, 2022. [Online]. Available: <https://dylanm.org/drafts/canonical-gradings.pdf>
- [9] R. Garner and I. López Franco, “Commutativity,” *Journal of Pure and Applied Algebra*, vol. 220, no. 5, pp. 1707–1751, 2016. [Online]. Available: <https://doi.org/10.1016/j.jpaa.2015.09.003>
- [10] R. Poklewski-Koziell, “Commutative graded monads,” 2022, working paper or preprint. [Online]. Available: <https://arxiv.org/abs/2204.01634>
- [11] D. Orchard, P. Wadler, and H. E. III, “Unifying graded and parameterised monads,” in *Proceedings Eighth Workshop on Mathematically Structured Functional Programming, MSFP ETAPS 2020, Dublin, Ireland, 25th April 2020*, ser. EPTCS, M. S. New and S. Lindley, Eds., vol. 317, 2020, pp. 18–38. [Online]. Available: <https://doi.org/10.4204/EPTCS.317.2>
- [12] B. Jacobs, *Introduction to Coalgebra: Towards Mathematics of States and Observation*, ser. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2016, vol. 59. [Online]. Available: <https://doi.org/10.1017/CBO9781316823187>

APPENDIX

A. Alternative Definitions for Graded Monads

Definition 30 (Pomonoid Graded Monads). If we change the monoid as a pomonoid : $\mathcal{G} : ((G, \leq), i, *)$ as grades, $a, b \in G$. Here G is equipped with a partial order \leq , and $*$ is monotone. We define some extra rules:

- for any $a \leq a', b \leq b'$ in \mathcal{G} , a natural transformation $\overset{a \leq a'}{\mathcal{T}} : \overset{a}{\mathcal{T}} \rightarrow \overset{a'}{\mathcal{T}}$.

s.t.

$$\overset{a \leq a}{\mathcal{T}} = I_{\overset{a}{\mathcal{T}}}$$

$$\overset{a' \leq a''}{\mathcal{T}} \circ \overset{a \leq a'}{\mathcal{T}} = \overset{a \leq a' \leq a''}{\mathcal{T}}$$

$$\begin{array}{ccc} \overset{a}{\mathcal{T}} \cdot \overset{b}{\mathcal{T}} & \xrightarrow{\mu^{a,b}} & \overset{a*b}{\mathcal{T}} \\ \downarrow \overset{a \leq a' \quad b \leq b'}{\mathcal{T}} & & \downarrow \overset{a*b \leq a'*b'}{\mathcal{T}} \\ \overset{a'}{\mathcal{T}} \cdot \overset{b'}{\mathcal{T}} & \xrightarrow{\mu^{a',b'}} & \overset{a'*b'}{\mathcal{T}} \end{array}$$

Now it is a lax functor from a thin monoidal category.

B. Lemmas for proof of Theorem 17 and 24

Lemma 31. If $(X, f : X \rightarrow \overset{\phi a}{\mathcal{T}}Y)$ is a graded central cone of \mathcal{T} at (a, Y) . Then for any $g : Z \rightarrow X$ in \mathbf{C} , it follows that $(Z, f \circ g)$ is a graded central cone of \mathcal{T} at (a, Y) . (Remark: If we take \mathcal{G}' as a terminal monoidal category, we have a normal monad.)

Proof. This is obtained by precomposing the definition of graded central cone by $g \otimes \text{id}$. For all $b' \in G'$ and X' in \mathbf{C} ,

$$\begin{array}{ccccccc}
Z \otimes \overset{b'}{\mathcal{T}}X' & \xrightarrow{g \otimes \overset{b'}{\mathcal{T}}X'} & X \otimes \overset{b'}{\mathcal{T}}X' & \xrightarrow{f \otimes \overset{b'}{\mathcal{T}}X'} & \overset{\phi a}{\mathcal{T}}Y \otimes \overset{b'}{\mathcal{T}}X' & \xrightarrow{\tau'_{Y, \overset{b'}{\mathcal{T}}X'}} & \overset{\phi a}{\mathcal{T}}(Y \otimes \overset{b'}{\mathcal{T}}X') \\
& & \downarrow f \otimes \overset{b'}{\mathcal{T}}X' & & & & \downarrow \overset{\phi a}{\mathcal{T}}\tau_{Y, X'} \\
& & \overset{\phi a}{\mathcal{T}}Y \otimes \overset{b'}{\mathcal{T}}X' & & & & \overset{\phi a}{\mathcal{T}} \cdot \overset{b'}{\mathcal{T}}(Y \otimes X') \\
& & \downarrow \tau_{\overset{\phi a}{\mathcal{T}}Y, X'} & & & & \downarrow \mu_{Y \otimes X'}^{\phi a, b'} \\
& & \overset{b'}{\mathcal{T}}(\overset{\phi a}{\mathcal{T}}Y \otimes X') & \xrightarrow{\overset{b'}{\mathcal{T}}\tau'_{Y, X'}} & \overset{b'}{\mathcal{T}} \cdot \overset{\phi a}{\mathcal{T}}(Y \otimes X') & \xrightarrow{\mu_{Y \otimes X'}^{b', \phi a}} & \overset{b' \otimes \phi a}{\mathcal{T}}(Y \otimes X') = \overset{\phi a \otimes b'}{\mathcal{T}}(Y \otimes X')
\end{array}$$

commutes directly from the definition of graded central cone for f . \square

Lemma 32. If $(X, f : X \rightarrow \overset{\phi a}{\mathcal{T}}Y)$ is a graded central cone of \mathcal{T} at (a, Y) . Then for any $g : Y \rightarrow Z$ in \mathbf{C} at a , it follows that $(X, \overset{\phi a}{\mathcal{T}}g \circ f)$ is a graded central cone of \mathcal{T} at Z at a . (Remark: If we take \mathcal{G}' as a terminal monoidal category, we have a normal monad.)

Proof. The naturality of τ and μ allow us to push the application of g to the last postcomposition, in order to use the central property of f . In more details, for all $b' \in G'$ and X' in \mathbf{C} , the following diagram:

$$\begin{array}{ccccccc}
X \otimes \overset{b'}{\mathcal{T}}X' & \xrightarrow{f \otimes \overset{b'}{\mathcal{T}}X'} & \overset{\phi a}{\mathcal{T}}Y \otimes \overset{b'}{\mathcal{T}}X' & \xrightarrow{\overset{\phi a}{\mathcal{T}}g \otimes \overset{b'}{\mathcal{T}}X'} & \overset{\phi a}{\mathcal{T}}Z \otimes \overset{b'}{\mathcal{T}}X' & & \\
\downarrow f \otimes \overset{b'}{\mathcal{T}}X' & & \downarrow \tau'_{Y, \overset{b'}{\mathcal{T}}X'} & \text{(2)} & \downarrow \tau'_{Z, \overset{b'}{\mathcal{T}}X'} & & \\
\overset{\phi a}{\mathcal{T}}Y \otimes \overset{b'}{\mathcal{T}}X' & \xrightarrow{\overset{\phi a}{\mathcal{T}}\tau_{Y, X'}} & \overset{\phi a}{\mathcal{T}}(Y \otimes X') & \xrightarrow{\overset{\phi a}{\mathcal{T}}(g \otimes \overset{b'}{\mathcal{T}}X')} & \overset{\phi a}{\mathcal{T}}(Z \otimes X') & & \\
\downarrow \overset{\phi a}{\mathcal{T}}\tau_{Y, X'} & & \downarrow \mu_{Y \otimes X'}^{\phi a, b'} & \text{(3)} & \downarrow \mu_{Z \otimes X'}^{\phi a, b'} & & \\
\overset{\phi a}{\mathcal{T}} \cdot \overset{b'}{\mathcal{T}}(Y \otimes X') & \xrightarrow{\overset{a}{\mathcal{T}} \cdot \overset{b}{\mathcal{T}}(g \otimes X')} & \overset{\phi a}{\mathcal{T}} \cdot \overset{b'}{\mathcal{T}}(Z \otimes X') & & & & \\
\downarrow \mu_{Y \otimes X'}^{\phi a, b'} & & \downarrow \mu_{Z \otimes X'}^{\phi a, b'} & \text{(4)} & & & \\
\overset{\phi a}{\mathcal{T}}Z \otimes \overset{b'}{\mathcal{T}}X' & \xrightarrow{\tau_{\overset{\phi a}{\mathcal{T}}Z, X'}} & \overset{b'}{\mathcal{T}}(\overset{\phi a}{\mathcal{T}}Z \otimes X') & \xrightarrow{\overset{b'}{\mathcal{T}}\tau'_{Z, X'}} & \overset{b'}{\mathcal{T}} \cdot \overset{\phi a}{\mathcal{T}}(Z \otimes X') & \xrightarrow{\mu_{Z \otimes X'}^{b', \phi a}} & \overset{b' \otimes \phi a}{\mathcal{T}}(Z \otimes X') = \overset{\phi a \otimes b'}{\mathcal{T}}(Z \otimes X') \\
\downarrow \overset{\phi a}{\mathcal{T}}g \otimes \overset{b'}{\mathcal{T}}X' & \text{(5)} & \downarrow \overset{b'}{\mathcal{T}}(\overset{\phi a}{\mathcal{T}}g \otimes X') & \text{(6)} & \downarrow \overset{b'}{\mathcal{T}} \cdot \overset{\phi a}{\mathcal{T}}(g \otimes X') & \text{(7)} & \downarrow \overset{b' \otimes \phi a}{\mathcal{T}}(g \otimes X') & \text{(8)}
\end{array}$$

commutes, because: (1) f is a graded central cone, (2) τ' is natural, (3) τ is natural, (4) μ is natural (5) τ is natural, (6) τ' is natural, (7) μ is natural, (8) \mathcal{T} is a functor. \square

Lemma 33. If (Z, ι) is a terminal (graded) central cone of \mathcal{T} at X , then ι is a monomorphism.

Proof. Let us consider $f, g : Y \rightarrow Z$ such that $\iota \circ f = \iota \circ g$; this family of morphism is a graded central cone at X (Lemma 31), and since (Z, ι) is a terminal graded central cone, it factors uniquely through ι . Thus $f = g$ and therefore ι is monic. \square

Lemma 34. For $A := (W \otimes \mathcal{T}X) \otimes Y$

$$\tau'_{W \otimes X, Y} \circ \tau_{W, X} \otimes Y \circ A = \mathcal{T}\alpha_{W, X, Y}^{-1} \circ \tau_{W, X \otimes Y} \circ W \otimes \tau'_{X, Y} \circ \alpha_{W, \mathcal{T}X, Y} \circ A$$

Proof. Left = $\mathcal{T}(W \otimes X) \otimes Y$ = Right. \square

C. Parts of proof of Theorem 17

1) Part I of proof of Theorem 17:

Proof. First part:

Recall that \mathcal{Z} maps every object X to its terminal central cone at X . Let $f : X \rightarrow Y$ be a morphism. $\mathcal{T}f \circ \iota_X : \mathcal{Z}X \rightarrow \mathcal{T}Y$ is a central cone according to Lemma 32. Therefore, by proposition 15, we can define $\mathcal{Z}f$ as the unique map such that the following diagram commutes:

$$\begin{array}{ccc} & \mathcal{Z}f & \\ \mathcal{Z}X & \dashrightarrow & \mathcal{Z}Y \\ \iota_X \downarrow & & \downarrow \iota_Y \\ \mathcal{T}X & \xrightarrow{\mathcal{T}f} & \mathcal{T}Y \end{array}$$

It follows directly that \mathcal{Z} maps the identity to the identity, and that ι is natural. \mathcal{Z} also preserves composition, which follows by the commutative diagram below:

$$\begin{array}{ccc} & \iota_W & \\ & \mathcal{Z}W \xrightarrow{\quad} \mathcal{T}W & \\ \mathcal{Z}g \downarrow & & \downarrow \mathcal{T}g \\ & \iota_X & \\ \mathcal{Z}X & \xrightarrow{\quad} \mathcal{T}X & \\ \mathcal{Z}f \downarrow & & \downarrow \mathcal{T}f \\ & \iota_Y & \\ & \mathcal{Z}Y \xrightarrow{\quad} \mathcal{T}Y & \end{array} \quad \begin{array}{l} \mathcal{Z}(f \circ g) \\ \mathcal{T}(f \circ g) \end{array}$$

This proves that \mathcal{Z} is a functor. □

2) Rest of proof of Theorem 17:

Proof. Rest parts of Part two : Next we proof that they are natural:

Using the fact that ι is monic (see Lemma 33) we show that the following commutative diagram shows that $\eta^{\mathcal{Z}}$ is natural.

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X^{\mathcal{Z}}} & \mathcal{T}X & \xrightarrow{\iota_X} & \mathcal{Z}X \\ & \searrow \eta_X & & \swarrow \mathcal{Z}f & \\ & & \mathcal{T}X & & \mathcal{Z}Y \\ & & \searrow \mathcal{T}f & & \downarrow \iota_Y \\ Y & \xrightarrow{\eta_Y^{\mathcal{Z}}} & \mathcal{T}Y & \xrightarrow{\iota_Y} & \mathcal{T}Y \end{array}$$

(1) (2) (3) (4)

(1) definition of $\eta^{\mathcal{Z}}$, (2) ι is natural, (3) η is natural and (4) definition of $\eta^{\mathcal{Z}}$. Thus, we have proven that for any $f : X \rightarrow Y$, $\iota_Y \circ \mathcal{Z}f \circ \eta_X^{\mathcal{Z}} = \iota_Y \circ \eta_Y^{\mathcal{Z}} \circ f$. Besides, ι is monic, thus $\mathcal{Z}f \circ \eta_X^{\mathcal{Z}} = \eta_Y^{\mathcal{Z}} \circ f$ which proves that $\eta^{\mathcal{Z}}$ is natural. We will prove all the remaining diagrams with the same reasoning.

The following commutative diagram shows that $\mu^{\mathcal{Z}}$ is natural.

$$\begin{array}{ccccc} \mathcal{Z}^2 X & \xrightarrow{\mu_X^{\mathcal{Z}}} & \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{T}X \\ & \searrow u & & \swarrow \mathcal{Z}f & \\ & & \mathcal{T}^2 X & & \mathcal{Z}Y \\ & & \searrow \mathcal{T}^2 f & & \downarrow \iota_Y \\ \mathcal{Z}^2 Y & \xrightarrow{\mu_Y^{\mathcal{Z}}} & \mathcal{T}^2 Y & \xrightarrow{\mu_Y} & \mathcal{T}Y \end{array}$$

(1) (2) (3) (4) (5)

(1) definition of $\mu^{\mathcal{Z}}$, (2) ι is natural, (3) μ is natural, (4) ι is natural and (5) definition of $\mu^{\mathcal{Z}}$.
The following commutative diagrams shows that $\tau^{\mathcal{Z}}$ is natural.

$$\begin{array}{ccccc}
 A \otimes \mathcal{Z}C & \xrightarrow{\tau_{A,C}^{\mathcal{Z}}} & & \mathcal{Z}(A \otimes C) & \\
 \downarrow f \otimes \mathcal{Z}C & \searrow A \otimes \iota & \xrightarrow{\tau_{A,C}^{\mathcal{Z}}} & \mathcal{Z}(f \otimes C) & \downarrow \iota \\
 & & A \otimes \mathcal{T}C & \xrightarrow{\tau_{A,C}} & \mathcal{T}(A \otimes C) & \xrightarrow{\iota} & \mathcal{Z}(A \otimes C) \\
 & & \downarrow f \otimes \mathcal{T}C & & \downarrow \mathcal{T}(f \otimes C) & & \downarrow \iota \\
 & & B \otimes \mathcal{T}C & \xrightarrow{\tau_{B,C}} & \mathcal{T}(B \otimes C) & \xrightarrow{\iota_{B \otimes C}} & \mathcal{Z}(B \otimes C) \\
 & & \downarrow \tau_{B,C} & & \downarrow \tau_{B,C} & & \downarrow \iota_{B \otimes C} \\
 B \otimes \mathcal{Z}C & \xrightarrow{\tau_{B,C}^{\mathcal{Z}}} & \mathcal{Z}(B \otimes C) & \xrightarrow{\iota_{B \otimes C}} & \mathcal{T}(B \otimes C) & & \mathcal{Z}(B \otimes C)
 \end{array}$$

(1) definition of $\tau^{\mathcal{Z}}$, (2) ι is natural, (3) τ is natural, (4) ι is natural and (5) definition of $\tau^{\mathcal{Z}}$.

$$\begin{array}{ccccc}
 A \otimes \mathcal{Z}B & \xrightarrow{\tau_{A,B}^{\mathcal{Z}}} & & \mathcal{Z}(A \otimes B) & \\
 \downarrow A \otimes \mathcal{Z}f & \searrow A \otimes \iota & \xrightarrow{\tau_{A,B}^{\mathcal{Z}}} & \mathcal{Z}(A \otimes f) & \downarrow \iota \\
 & & A \otimes \mathcal{T}B & \xrightarrow{\tau_{A,B}} & \mathcal{T}(A \otimes B) & \xrightarrow{\iota} & \mathcal{Z}(A \otimes B) \\
 & & \downarrow A \otimes \mathcal{T}f & & \downarrow \mathcal{T}(A \otimes f) & & \downarrow \iota \\
 & & A \otimes \mathcal{T}C & \xrightarrow{\tau_{A,C}} & \mathcal{T}(A \otimes C) & \xrightarrow{\iota_{A \otimes C}} & \mathcal{Z}(A \otimes C) \\
 & & \downarrow \tau_{A,C} & & \downarrow \tau_{A,C} & & \downarrow \iota_{A \otimes C} \\
 A \otimes \mathcal{Z}C & \xrightarrow{\tau_{A,C}^{\mathcal{Z}}} & \mathcal{Z}(A \otimes C) & \xrightarrow{\iota_{A \otimes C}} & \mathcal{T}(A \otimes C) & & \mathcal{Z}(A \otimes C)
 \end{array}$$

(1) definition of $\tau^{\mathcal{Z}}$, (2) ι is natural, (3) τ is natural, (4) ι is natural and (5) definition of $\tau^{\mathcal{Z}}$.
The following commutative diagrams prove that \mathcal{Z} is a monad.

$$\begin{array}{ccccc}
 \mathcal{Z}^3 X & \xrightarrow{\mu_{\mathcal{Z}^2 X}^{\mathcal{Z}}} & & \mathcal{Z}^2 X & \\
 \downarrow \mathcal{Z} \mu_X^{\mathcal{Z}} & \searrow \iota^3 & \xrightarrow{\mu_{\mathcal{T}^2 X}^{\mathcal{Z}}} & \mathcal{Z}^2 X & \downarrow \mu_X^{\mathcal{Z}} \\
 & & \mathcal{T}^3 X & \xrightarrow{\mu_{\mathcal{T}^2 X}} & \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{Z} X \\
 & & \downarrow \mathcal{T} \mu_X & & \downarrow \mu_X & & \downarrow \iota_X \\
 & & \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{T} X & \xrightarrow{\iota_X} & \mathcal{Z} X \\
 & & \downarrow \mu_X & & \downarrow \mu_X & & \downarrow \iota_X \\
 \mathcal{Z}^2 X & \xrightarrow{\mu_X^{\mathcal{Z}}} & \mathcal{Z} X & \xrightarrow{\iota_X} & \mathcal{T} X & & \mathcal{Z} X
 \end{array}$$

$$\begin{array}{ccccc}
 \mathcal{Z} X & \xrightarrow{\eta_{\mathcal{Z}^2 X}^{\mathcal{Z}}} & & \mathcal{Z}^2 X & \\
 \downarrow \mathcal{Z} \eta_X^{\mathcal{Z}} & \searrow \iota & \xrightarrow{\eta_{\mathcal{T}^2 X}^{\mathcal{Z}}} & \mathcal{Z}^2 X & \downarrow \mu_X^{\mathcal{Z}} \\
 & & \mathcal{T} X & \xrightarrow{\eta_{\mathcal{T}^2 X}} & \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{Z} X \\
 & & \downarrow \mathcal{T} \eta_X & & \downarrow \mu_X & & \downarrow \iota_X \\
 & & \mathcal{T}^2 X & \xrightarrow{\mu_X} & \mathcal{T} X & \xrightarrow{\iota_X} & \mathcal{Z} X \\
 & & \downarrow \mu_X & & \downarrow \mu_X & & \downarrow \iota_X \\
 \mathcal{Z}^2 X & \xrightarrow{\mu_X^{\mathcal{Z}}} & \mathcal{Z} X & \xrightarrow{\iota_X} & \mathcal{T} X & & \mathcal{Z} X
 \end{array}$$

(1) and (2) involve the definition of $\mu^{\mathcal{Z}}$ and the naturality of ι and $\mu^{\mathcal{Z}}$, (3) is by definition of monad, (4) definition of $\mu^{\mathcal{Z}}$ and (5) also. (6) and (7) involve the definition of $\eta^{\mathcal{Z}}$ and the naturality of ι and $\eta^{\mathcal{Z}}$, (8) is by definition of monad, (9) definition of $\mu^{\mathcal{Z}}$ and (10) also.

Last part:

\mathcal{Z} is proven strong with very similar diagrams.

The following commutative diagram proves that \mathcal{Z} is a commutative monad:

$$\begin{array}{ccccc}
 ZX \otimes ZY & \xrightarrow{\tau'^{\mathcal{Z}}_{X,ZY}} & Z(X \otimes ZY) & \xrightarrow{\mathcal{Z}\tau'^{\mathcal{Z}}_{X,Y}} & Z^2(X \otimes Y) \\
 \downarrow \tau^{\mathcal{Z}}_{ZX,Y} & \searrow \iota & \downarrow \iota & \searrow \iota & \downarrow \mu^{\mathcal{Z}}_{X \otimes Y} \\
 & & ZX \otimes TY & \xrightarrow{\tau'^{\mathcal{Z}}_{X,TY}} & Z(X \otimes TY) & \xrightarrow{\mathcal{Z}\tau_{X,Y}} & Z\mathcal{T}(X \otimes Y) \\
 & & \downarrow \iota & & \downarrow \iota & & \downarrow \iota \\
 & & TX \otimes ZY & \xrightarrow{\tau^{\mathcal{Z}}_{TX,ZY}} & TX \otimes TY & \xrightarrow{\tau'_{X,TY}} & \mathcal{T}(X \otimes TY) & \xrightarrow{\mathcal{T}\tau_{X,Y}} & \mathcal{T}^2(X \otimes Y) \\
 & & \downarrow \tau^{\mathcal{Z}}_{TX,Y} & & \downarrow \tau_{TX,Y} & & \downarrow \mu_{X \otimes Y} & & \downarrow \iota_{X \otimes Y} \\
 & & Z(ZX \otimes Y) & \xrightarrow{\iota} & Z(\mathcal{T}X \otimes Y) & \xrightarrow{\iota} & \mathcal{T}(\mathcal{T}X \otimes Y) & & \downarrow \mu_{X \otimes Y} \\
 & & \downarrow \mathcal{Z}\tau'^{\mathcal{Z}}_{X,Y} & & \downarrow \mathcal{Z}\tau'_{X,Y} & & \downarrow \mathcal{T}\tau'_{X,Y} & & \downarrow \mu_{X \otimes Y} \\
 & & Z^2(X \otimes Y) & \xrightarrow{\iota} & Z\mathcal{T}(X \otimes Y) & \xrightarrow{\iota} & \mathcal{T}^2(X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & \mathcal{T}(X \otimes Y) \\
 & & \downarrow \mu^{\mathcal{Z}}_{X \otimes Y} & & \downarrow \mu^{\mathcal{Z}}_{X \otimes Y} & & \downarrow \mu_{X \otimes Y} & & \downarrow \iota_{X \otimes Y} \\
 & & ZX \otimes Y & \xrightarrow{\iota} & Z(X \otimes Y) & \xrightarrow{\iota_{X \otimes Y}} & \mathcal{T}(X \otimes Y) & &
 \end{array}
 \tag{2}$$

(1) $\tau'^{\mathcal{Z}}$ is natural, (2) definition of $\tau^{\mathcal{Z}}$, (3) $\tau^{\mathcal{Z}}$ is natural, (4) \mathbf{C} is monoidal, (5) definition of $\tau'^{\mathcal{Z}}$, (6) ι is natural, (7) definition of $\mu^{\mathcal{Z}}$, (8) definition of $\tau^{\mathcal{Z}}$, (9) ι is central, (10) definition of $\tau'^{\mathcal{Z}}$, (11) ι is natural and (12) definition of $\mu^{\mathcal{Z}}$. \square

D. Kleisli structure and Premonoidal Structure of Strong Monads

Definition 35 (Kleisli category). Given a monad (\mathcal{T}, η, μ) over a category \mathbf{C} , the *Kleisli category* $\mathbf{C}_{\mathcal{T}}$ of \mathcal{T} is the category whose objects are the same as those of \mathbf{C} , but whose morphisms are given by $\mathbf{C}_{\mathcal{T}}[X, Y] = \mathbf{C}[X, \mathcal{T}Y]$. Composition in $\mathbf{C}_{\mathcal{T}}$ is given by $g \odot f \stackrel{\text{def}}{=} \mu_Z \circ \mathcal{T}g \circ f$ where $f : X \rightarrow \mathcal{T}Y$ and $g : Y \rightarrow \mathcal{T}Z$. The identity at X is given by the monadic unit $\eta_X : X \rightarrow \mathcal{T}X$.

Definition 36 (Premonoidal Centre [3]). Given a strong monad $(\mathcal{T}, \eta, \mu, \tau)$ on a symmetric monoidal category (\mathbf{C}, I, \otimes) , we say that a morphism $f : X \rightarrow Y$ in $\mathbf{C}_{\mathcal{T}}$ is *central* if for any morphism $f' : X' \rightarrow Y'$ in $\mathbf{C}_{\mathcal{T}}$, the diagram

$$\begin{array}{ccc}
 X \otimes X' & \xrightarrow{f \otimes_I X'} & Y \otimes X' \\
 X \otimes_r f' \downarrow & & \downarrow Y \otimes_r f' \\
 X \otimes Y' & \xrightarrow{f \otimes_I Y'} & Y \otimes Y'
 \end{array}$$

commutes in $\mathbf{C}_{\mathcal{T}}$. The *premonoidal centre* of $\mathbf{C}_{\mathcal{T}}$ is the subcategory $Z(\mathbf{C}_{\mathcal{T}})$ which has the same objects as those of $\mathbf{C}_{\mathcal{T}}$ and whose morphisms are the central morphisms of $\mathbf{C}_{\mathcal{T}}$.